Spectral properties of the Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^N, \mu)$ for $1 \leq p < \infty$

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Let us consider the Ornstein-Uhlenbeck operator

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i,j=1}^{N} b_{ij} x_j D_i = \mathrm{Tr}(QD^2) + \langle Bx, S \rangle, \quad x \in \mathbb{R}^N, \tag{1}$$

where $Q = (q_{ij})$ is a real, symmetric and non-negative matrix and $B = (b_{ij})$ is a non-zero real matrix. The associated Markov semigroup $(T(t))_{t\geq 0}$ has the following representation due to Kolmogoroff

$$(T(t)f)(x) = \frac{1}{(4\pi)^{N/2}(\mathrm{det}Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\langle Q_t^{-1}y,y\rangle/4} f(e^{tB}x - y), \ t > 0, \ x \in \mathbb{R}^N, \tag{2}$$

for $f \in C_b(\mathbb{R}^N)$, where Q_t is defined by

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} \, ds, \quad t > 0,$$

and B^* denotes the adjoint matrix of B, [1]. We suppose that $\mathrm{det}Q_t > 0$ for any $t > 0$ and that the spectrum of B is contained in $\mathbb{C}^- = \{\lambda \in \mathbb{C} : \ \mathrm{Re}\lambda < 0\}$.

Recall that under the assumption $\mathrm{det}Q_t > 0$ for any $t > 0$, the condition $\sigma(B) \subset \mathbb{C}^-$ is equivalent to the existence of an invariant measure μ for $(T(t))_{t\geq 0}$, that is, the existence of a probability measure μ on $\mathbb{R}^N$ satisfying

$$\int_{\mathbb{R}^N} (T(t)f)(x) d\mu(x) = \int_{\mathbb{R}^N} f(x) d\mu(x)$$

for all $t \geq 0$ and $f \in C_b(\mathbb{R}^N)$, [2, Section 11.2.3]. In particular, the invariant measure μ is unique and given by $d\mu(x) = \rho(x)dx$, where

$$\rho(x) = \frac{1}{(4\pi)^{N/2}(\mathrm{det}Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}y,y\rangle/4}, \quad x \in \mathbb{R}^N, \tag{3}$$

and $Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*} \, ds$.

The semigroup $(T(t))_{t\geq 0}$ extends to a strongly continuous semigroup of positive contractions in $L^p(\mathbb{R}^N, d\mu)$ for every $1 \leq p < \infty$. We observe that the fact that $Q_t < Q_\infty$ in the sense of quadratic forms implies that the integral in (2) converges for every $f \in L^p(\mathbb{R}^N, d\mu)$, $t > 0$ and $x \in \mathbb{R}^N$. So, the extension of $(T(t))_{t\geq 0}$ to $L^p(\mathbb{R}^N, d\mu)$, $1 \leq p < \infty$, is still given by formula (2).

Let us denote by (A_p, D_p) the generator of $(T(t))_{t\geq 0}$ in $L^p(\mathbb{R}^N, d\mu)$, $1 \leq p < \infty$. The aim of this project is to describe the spectrum of (A_p, D_p) for $1 \leq p < \infty$. In particular, supposing that $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of B, we show for $1 < p < \infty$ that $\sigma(A_p) = \sigma_{pt}(A_p) = \{\sum_{i=1}^r n_i\lambda_i : \ n_i \in \mathbb{N}\}$ and that all generalized eigenfunctions are polynomials which form a complete system in $L^p(\mathbb{R}^N, d\mu)$. We also show that in case $p = 1$ the spectrum is completely different, that is, $\sigma(A_1) = \overline{\mathbb{C}^-}$ and $\sigma_{pt}(A_1) = \mathbb{C}^-$. The main reference is [3].

References

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