EXERCISES FOR LECTURE 9

SOLUTIONS PROVIDED BY THE KONSTANZ TEAM

1. Exercise 9.1 (Solution by Max Nendel)

We are asked to prove Lemma 9.1.1:
For every $f \in C_b^1(\mathbb{R}^d)$ and for every $i = 1, \dots, d$ we have

$$\int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}(x)\gamma_d(dx) = \int_{\mathbb{R}^d} x_i f(x)\, \gamma_d(dx).$$

Without loss of generality let $i = d$. Let $r > 0$ and $x' \in \mathbb{R}^{d-1}$. Then, using integration by parts we obtain that

$$\begin{aligned}\int_{-r}^{r} \frac{\partial f}{\partial x_d}(x', x_d) e^{-(|x'|^2+|x_d|^2)/2} dx_d &= -f(x', x_d) e^{-(|x'|^2+|x_d|^2)/2}|_{x_d=-r}^{r} \\ &\quad + \int_{-r}^{r} f(x', x_d) x_d e^{-(|x'|^2+|x_d|^2)/2} dx \\ &\to \int_{-r}^{r} f(x', x_d) x_d e^{-(|x'|^2+|x_d|^2)/2} dx_d\end{aligned}$$

as $r \to \infty$. By the dominated convergence theorem, we thus obtain that

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x_d}(x', x_d) e^{-(|x'|^2+|x_d|^2)/2} dx_d = \int_{\mathbb{R}} f(x', x_d) x_d e^{-(|x'|^2+|x_d|^2)/2} dx_d.$$

For $x \in \mathbb{R}^d$ let $x' := (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$. Then, using Fubini's theorem, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\partial f}{\partial x_d}(x)\gamma_d(dx) &= \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_d}(x) e^{-|x|^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{\partial f}{\partial x_d}(x', x_d) e^{-(|x'|^2+|x_d|^2)/2} dx_d dx' \\ &= \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} f(x', x_d) x_d e^{-|x|^2/2} dx_d dx' \\ &= \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} x_d f(x) e^{-|x|^2/2} dx \\ &= \int_{\mathbb{R}^d} x_d f(x)\gamma_d(dx).\end{aligned}$$

2. Exercise 9.2 (Solution by Asgar Jamneshan and Markus Kunze)

Part (v) of Proposition 9.1.5 reads as follows:

Let $1 < p < \infty$. If $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $\sup_{n\in\mathbb{N}} \|f_n\|_{W^{1,1}(\mathbb{R}^d,\gamma_d)} < \infty$, then $f \in W^{1,p}(\mathbb{R}^d, \gamma_d)$.

In this exercise we are asked to prove that the corresponding statement for $p = 1$ and $d = 1$ is wrong and to deduce from this that $W^{1,1}(\mathbb{R}^1, \gamma_1)$ is not reflexive.

(i) As in the hint, let $f_n(x) = 0$ for $x \le 0$, $f_n(x) = nx$ for $0 \le x \le 1/n$ and $f_n(x) = 1$ for $x \ge 1/n$. Let $f(x) = 0$ for $x \le 0$ and $f(x) = 1$ for $x > 0$. Then $|f_n| \le 1$ and f_n converges pointwise to f. By the dominated convergence theorem, $f_n \to f$ in $L^1(\mathbb{R}^1, \gamma_1)$. Moreover, $\sup_{n\in\mathbb{N}} \|f_n\|_{W^{1,1}(\mathbb{R}^1,\gamma_1)} \le 2$. However, f is not continuous, and thus $f \notin W^{1,1}_{\text{loc}}(\mathbb{R}^1)$. Using Proposition 9.1.6, it follows that $f \notin W^{1,1}(\mathbb{R}^1, \gamma_1)$.

(ii) Consider the functions f_n and f as in Part (i). Then the sequence f_n is bounded. If $W^{1,1}(\mathbb{R}^1, \gamma_1)$ was reflexive, we would find a subsequence converging weakly to some $g \in W^{1,1}(\mathbb{R}^1, \gamma_1)$. As $f_n \to f$ in $L^1(\mathbb{R}^1, \gamma_1)$, we would have $f = g$—a contradiction.

3. Exercise 9.3 (Solution by Asgar Jamneshan)

Given a measurable space $(\Omega, \mathscr{F})$ and a positive finite measure μ on it we are asked to prove that a function $f : \Omega \to \mathbb{R}$ is measurable if and only if it is the pointwise a.e. limit of a sequence of simple function.

Let us first point out that this statement need not be true if the measure space $(\Omega, \mathscr{F}, \mu)$ is not complete. Indeed, if we can find a non measurable nullset E, then $\mathbb{1}_E$ is the pointwise a.e. limit of the sequence $f_n \equiv 0$ of simple functions, yet it is not measurable.

In what follows we assume that $(\Omega, \mathscr{F}, \mu)$ is a finite and *complete* measure space.

Let $f : \Omega \to \mathbb{R}$ be a measurable function. Denote by $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then both f^+ and f^- are measurable functions and $f = f^+ - f^-$.

Recall that for a measurable function $h : \Omega \to \mathbb{R}$ with $h \ge 0$, the sequence (h_n) of simple functions defined by

$$\sum_{j=0}^{n2^n} j2^{-n} 1_{C_{j,n}}$$

where

$$C_{j,n} = \begin{cases} \{j2^{-n} \le f < (j+1)2^{-n}\}, & 0 \le j \le n2^n - 1, \\ \{f \ge n\}, & j = n2^n \end{cases}$$

converges *pointwise* to h. Let (g_n^+) be such an approximating sequence for f^+ and (g_n^-) for f^-. Since the difference of two converging sequences is convergent, f is the pointwise limit of the simple functions $f_n = g_n^+ - g_n^-$. In particular, f is the almost everywhere limit of the sequence (f_n).

On the other hand, recall that if $f = g$ almost everywhere (that is, $f = g$ on N^c where $\mu(N) = 0$) and g is measurable, then f is measurable whenever the underlying measure space is complete. Indeed, let I be a Borel subset of $\mathbb{R}$. We have

$$f^{-1}(I) = [f^{-1}(I) \cap N^c] \cup [f^{-1}(I) \cap N] = [g^{-1}(I) \cap N^c] \cup [f^{-1}(I) \cap N].$$

By assumption, $g^{-1}(I) \cap N^c \in \mathscr{F}$, and since $(\Omega, \mathscr{F}, \mu)$ is complete, $f^{-1}(I) \cap N \in \mathscr{F}$ as well. Now set $g = \liminf_{n\in\mathbb{N}} f_n$. Then g is measurable and $f = g$ almost everywhere because $f_n \to f$ almost everywhere.

4. Exercise 9.4 (Solution by Daniel Bartl)

In this exercise we should prove Lemma 9.3.6:
If $\psi \in L^1(X, \gamma)$ is such that

$$\int_X \psi\varphi \, d\gamma = 0 \quad \forall\, \varphi \in \mathcal{F}C_b^1(X),$$

then $\psi = 0$ a.e.

Define $\varphi := \mathbb{1}_{\{\psi\geq 0\}} - \mathbb{1}_{\{\psi<0\}}$ and observe that $\varphi \in L^\infty(X, \gamma)$. Thus, by Theorem 7.4.6 there exists a sequence (φ^n) in $\mathcal{F}C_b^\infty$ converging in $L^1(X, \gamma)$ to φ and we may assume (possibly after passing to a subsequence) γ-almost sure convergence. Further, for each n let η^n be the convolution of the function $x \mapsto \min\{1, \max\{-1, x\}\}$ and a mollifier with support $[-1/n, 1/n]$. Then the functions $\eta^n \circ \varphi^n$ are still in $\mathcal{F}C_b^\infty$ and converge to φ almost surely. Now we are able to apply the dominated convergence theorem and obtain $\int_X |\psi| d\gamma = \lim_{n\to\infty} \int_X \psi \cdot (\eta^n \circ \varphi^n) d\gamma = 0$ which shows that ψ equals 0 almost surly.