Lecture 9

Sobolev Spaces I

9.1 The finite dimensional case

We consider here the standard Gaussian measure $\gamma_d = \mathscr{N}(0, I_d)$ in $\mathbb{R}^d$. As in the case of the Lebesgue measure λ_d, for $1 \leq p < +\infty$ there are several equivalent definitions of the Sobolev space $W^{1,p}(\mathbb{R}^d, \gamma_d)$. It may be defined as the set of the functions in $L^p(\mathbb{R}^d, \gamma_d)$ having weak derivatives $D_i f$, $i = 1, \ldots, d$ in $L^p(\mathbb{R}^d, \gamma_d)$, or as the completion of a set of smooth functions in the Sobolev norm,

$$\|f\|_{W^{1,p}(\mathbb{R}^d,\gamma_d)} := \left(\int_{\mathbb{R}^d} |f|^p d\gamma_d \right)^{1/p} + \left(\int_{\mathbb{R}^d} |\nabla f|^p d\gamma_d \right)^{1/p}. \tag{9.1.1}$$

Such approaches are equivalent. We will follow the second one, which is easily extendable to the infinite dimensional case, and in the infinite dimensional case seems to be the simplest one. To begin with, we exhibit an integration formula for functions in $C_b^1(\mathbb{R}^d)$, the space of bounded continuously differentiable functions with bounded first order derivatives.

Lemma 9.1.1. *For every $f \in C_b^1(\mathbb{R}^d)$ and for every $i = 1, \ldots, d$ we have*

$$\int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}(x)\, \gamma_d(dx) = \int_{\mathbb{R}^d} x_i f(x)\, \gamma_d(dx). \tag{9.1.2}$$

The proof is left as an exercise. Applying Lemma 9.1.1 to the product fg we get the integration by parts formula

$$\int_{\mathbb{R}^d} f \frac{\partial g}{\partial x_i}\, d\gamma_d = -\int_{\mathbb{R}^d} g \frac{\partial f}{\partial x_i}\, d\gamma_d + \int_{\mathbb{R}^d} f(x)g(x)x_i\, \gamma_d(dx), \quad f,\ g \in C_b^1(\mathbb{R}^d), \tag{9.1.3}$$

which is the starting point of the theory of Sobolev spaces.

We recall the definition of a closable operator, and of the closure of a closable operator.

Definition 9.1.2. *Let E, F be Banach spaces and let $L : D(L) \subset E \to F$ be a linear operator. L is called* closable *(in E) if there exists a linear operator $\overline{L} : D(\overline{L}) \subset E \to F$ whose graph is the closure of the graph of L in $E \times F$. Equivalently, L is closable if*

$$(x_n) \subset D(L),\ \lim_{n\to\infty} x_n = 0 \text{ in } E,\ \lim_{n\to\infty} Lx_n = z \text{ in } F \Longrightarrow z = 0. \tag{9.1.4}$$

If L is closable, the domain of the closure $\overline{L}$ of L is the set

$$D(\overline{L}) = \Big\{x \in E :\ \exists (x_n) \subset D(L),\ \lim_{n\to\infty} x_n = x,\ \ Lx_n \text{ converges in } F\Big\}$$

and for $x \in D(\overline{L})$ we have

$$\overline{L}x = \lim_{n\to\infty} Lx_n,$$

for every sequence $(x_n) \subset D(L)$ such that $\lim_{n\to\infty} x_n = x$. Condition (9.1.4) guarantees that $\lim_{n\to\infty} Lx_n$ is independent of the sequence (x_n). Since $\overline{L}$ is a closed operator, its domain is a Banach space with the graph norm $x \mapsto \|x\|_E + \|\overline{L}x\|_F$.

For every $1 \le p < +\infty$ we set as usual $p' = p/(p-1)$ if $1 < p < +\infty$, $p' = +\infty$ if $p = 1$.

Lemma 9.1.3. *For any $1 \le p < +\infty$, the operator $\nabla : D(\nabla) = C^1_b(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$ is closable in $L^p(\mathbb{R}^d, \gamma_d)$.*

Proof. Let $f_n \in C^1_b(\mathbb{R}^d)$ be such that $f_n \to 0$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $\nabla f_n \to G = (g_1, \dots g_d)$ in $L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$. For every $i = 1, \dots, d$ and $\varphi \in C^1_c(\mathbb{R}^d)$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \frac{\partial f_n}{\partial x_i} \varphi \, d\gamma_d = \int_{\mathbb{R}^d} g_i \varphi \, d\gamma_d,$$

since

$$\int_{\mathbb{R}^d} \Big|\Big(\frac{\partial f_n}{\partial x_i} - g_i\Big)\varphi\Big| \, d\gamma_d \le \|\partial f_n/\partial x_i - g_i\|_{L^p(\mathbb{R}^d,\gamma_d)} \|\varphi\|_{L^{p'}(\mathbb{R}^d,\gamma_d)}.$$

On the other hand,

$$\int_{\mathbb{R}^d} \frac{\partial f_n}{\partial x_i} \varphi \, d\gamma_d = -\int_{\mathbb{R}^d} f_n \frac{\partial \varphi}{\partial x_i} \, d\gamma_d + \int_{\mathbb{R}^d} x_i f_n(x) \varphi(x) \, \gamma_d(dx), \quad n \in \mathbb{N},$$

so that, since $f_n \to 0$ in $L^p(\mathbb{R}^d, \gamma_d)$ and the functions $x \mapsto \partial\varphi/\partial x_i(x)$, $x \mapsto x_i \varphi(x)$ are bounded,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \frac{\partial f_n}{\partial x_i} \varphi \, d\gamma_d = 0.$$

So,

$$\int_{\mathbb{R}^d} g_i \, \varphi \, d\gamma_d = 0, \quad \varphi \in C^1_c(\mathbb{R}^d)$$

which implies $g_i = 0$ a.e. □

Lemma 9.1.3 allows to define the Sobolev spaces of order 1, as follows.

Definition 9.1.4. *For every $1 \le p < +\infty$, $W^{1,p}(\mathbb{R}^d, \gamma_d)$ is the domain of the closure of $\nabla : C_b^1(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$ in $L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$ (still denoted by ∇). Therefore, $f \in L^p(\mathbb{R}^d, \gamma_d)$ belongs to $W^{1,p}(\mathbb{R}^d, \gamma_d)$ iff there exists a sequence of functions $f_n \in C_b^1(\mathbb{R}^d)$ such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and ∇f_n converges in $L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$, and in this case, $\nabla f = \lim_{n\to\infty} \nabla f_n$. Moreover we set $\partial f/\partial x_i(x) := \nabla f(x) \cdot e_i$, $i = 1, \dots d$.*

$W^{1,p}(\mathbb{R}^d, \gamma_d)$ is a Banach space with the graph norm

$$\begin{aligned} \|f\|_{W^{1,p}(\mathbb{R}^d,\gamma_d)} &:= \|f\|_{L^p(\mathbb{R}^d,\gamma_d)} + \|\nabla f\|_{L^p(\mathbb{R}^d,\gamma_d;\mathbb{R}^d)} \\ &= \left(\int_{\mathbb{R}^d} |f|^p d\gamma_d\right)^{1/p} + \left(\int_{\mathbb{R}^d} |\nabla f|^p d\gamma_d\right)^{1/p}. \end{aligned} \tag{9.1.5}$$

One could give a more abstract definition of the Sobolev spaces, as the completion of $C_b^1(\mathbb{R}^d)$ in the norm (9.1.1). Since the norm (9.1.1) is stronger than the L^p norm, every element of the completion may be identified in an obvious way with an element f of $L^p(\mathbb{R}^d, \gamma_d)$. However, to define ∇f we need to know that for any sequence (f_n) of C_b^1 functions such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and ∇f_n is a Cauchy sequence in $L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$, the sequence of gradients (∇f_n) converges to the same limit in $L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$. In other words, we need Lemma 9.1.3.

Several properties of the spaces $W^{1,p}(\mathbb{R}^d, \gamma_d)$ follow easily.

Proposition 9.1.5. *Let $1 < p < +\infty$. Then*

(i) the integration formula (9.1.2) holds for every $f \in W^{1,p}(X, \gamma_d)$, $i = 1, \dots, d$;

(ii) if $\theta \in C_b^1(\mathbb{R}^d)$ and $f \in W^{1,p}(\mathbb{R}^d, \gamma_d)$, then $\theta \circ f \in W^{1,p}(\mathbb{R}^d, \gamma_d)$, and $\nabla(\theta \circ f) = (\theta' \circ f)\nabla f$;

(iii) if $f \in W^{1,p}(\mathbb{R}^d, \gamma_d)$, $g \in W^{1,q}(\mathbb{R}^d, \gamma_d)$ with $1/p + 1/q = 1/s \le 1$, then $fg \in W^{1,s}(\mathbb{R}^d, \gamma_d)$ and
$$\nabla(fg) = g\nabla f + f\nabla g;$$

(iv) $W^{1,p}(\mathbb{R}^d, \gamma_d)$ is reflexive;

(v) if $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $\sup_{n\in\mathbb{N}} \|f_n\|_{W^{1,p}(\mathbb{R}^d,\gamma_d)} < \infty$, then $f \in W^{1,p}(\mathbb{R}^d, \gamma_d)$.

Proof. Statement (i) follows just approximating f by a sequence of functions belonging to $C_b^1(\mathbb{R}^d)$, using (9.1.2) for every approximating function f_n and letting $n \to \infty$.

Statement (ii) follows approaching $\theta \circ f$ by $\theta \circ f_n$, if $f_n \in C_b^1(\mathbb{R}^d)$ is such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $\nabla f_n \to \nabla f$ in $L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$.

Statement (iii) follows easily from the definition, approaching fg by $f_n g_n$ if $f_n \in C_b^1(\mathbb{R}^d)$ are such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$, $\nabla f_n \to \nabla f$ in $L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$, $g_n \to g$, in $L^q(\mathbb{R}^d, \gamma_d)$, $\nabla g_n \to \nabla g$ in $L^q(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$.

The proof of (iv) is similar to the standard proof of the reflexivity of $W^{1,p}(\mathbb{R}^d, \lambda_d)$. The mapping $u \mapsto Tu = (u, \nabla u)$ is an isometry from $W^{1,p}(\mathbb{R}^d, \gamma_d)$ to the product space $E := L^p(\mathbb{R}^d, \gamma_d) \times L^p(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$, which implies that the range of T is closed in E. Now,

$L^p(\mathbb{R}^d,\gamma_d)$ and $L^p(\mathbb{R}^d,\gamma_d;\mathbb{R}^d)$ are reflexive so that E is reflexive, and $T(W^{1,p}(\mathbb{R}^d,\gamma_d))$ is reflexive too. Being isometric to a reflexive space, $W^{1,p}(\mathbb{R}^d,\gamma_d)$ is reflexive.

As a consequence of reflexivity, if a sequence (f_n) is bounded in $W^{1,p}(\mathbb{R}^d,\gamma_d)$ a subsequence f_{n_k} converges weakly to an element g of $W^{1,p}(\mathbb{R}^d,\gamma_d)$ as $k\to\infty$. Since $f_{n_k}\to f$ in $L^p(\mathbb{R}^d,\gamma_d)$, then $f=g$ and statement (v) is proved. □

Note that the argument of the proof of (ii) works as well for $p=1$, and statement (ii) is in fact true also for $p=1$. Even statement (i) holds for $p=1$, but the fact that $x\mapsto x_if(x)\in L^1(\mathbb{R}^d,\gamma_d)$ for every $f\in W^{1,1}(\mathbb{R}^d,\gamma_d)$ is not obvious, and will not be considered in these lectures.

Instead, $W^{1,1}(\mathbb{R}^d,\gamma_d)$ is not reflexive, and statement (v) does not hold for $p=1$ (see Exercise 9.2).

The next characterisation is useful to recognise whether a given function belongs to $W^{1,p}(\mathbb{R}^d,\gamma_d)$. We recall that $L^p_{loc}(\mathbb{R}^d)$ (resp. $W^{1,p}_{loc}(\mathbb{R}^d)$) is the space of all (equivalence classes of) functions f such that the restriction of f to any ball B belongs to $L^p(B,\lambda_d)$ (resp. $W^{1,p}(B,\lambda_d)$). Equivalently, $f\in L^p_{loc}(\mathbb{R}^d)$ (resp. $f\in W^{1,p}_{loc}(\mathbb{R}^d)$) if $f\theta\in L^p(\mathbb{R}^d,\lambda_d)$ (resp. $f\theta\in W^{1,p}(\mathbb{R}^d,\lambda_d)$) for every $\theta\in C^\infty_c(\mathbb{R}^d)$. For $f\in W^{1,p}_{loc}(\mathbb{R}^d)$ we denote by D_if the weak derivative of f with respect to x_i, $i=1,\dots d$.

Proposition 9.1.6. *For every $1\le p<+\infty$,*

$$W^{1,p}(\mathbb{R}^d,\gamma_d)=\Big\{f\in W^{1,p}_{loc}(\mathbb{R}^d):\ f,\ D_if\in L^p(\mathbb{R}^d,\gamma_d),\ i=1,\dots d\Big\}.$$

Moreover, for every $f\in W^{1,p}(\mathbb{R}^d,\gamma_d)$ and $i=1,\dots,d$, $\partial f/\partial x_i$ coincides with the weak derivative D_if.

Proof. Let $f\in W^{1,p}(\mathbb{R}^d,\gamma_d)$. Then for every $g\in C^1_c(\mathbb{R}^d)$, (9.1.3) still holds: indeed, it is sufficient to approximate f by a sequence of functions belonging to $C^1_b(\mathbb{R}^d)$, to use (9.1.3) for every approximating function f_n, and to let $n\to\infty$.

This implies that $\partial f/\partial x_i$ is equal to the weak derivative D_if. Indeed, for every $\varphi\in C^\infty_c(\mathbb{R}^d)$, setting $g(x)=\varphi(x)e^{|x|^2/2}(2\pi)^{d/2}$, (9.1.3) yields

$$\int_{\mathbb{R}^d} f\frac{\partial\varphi}{\partial x_i}\,dx=\int_{\mathbb{R}^d} f\left(\frac{\partial g}{\partial x_i}-x_ig\right)d\gamma_d=-\int_{\mathbb{R}^d}\frac{\partial f}{\partial x_i}g\,d\gamma_d=-\int_{\mathbb{R}^d}\frac{\partial f}{\partial x_i}\varphi\,dx.$$

So, $\partial f/\partial x_i=D_if$, for every $i=1,\dots,d$. Since $L^p(\mathbb{R}^d,\gamma_d)\subset L^p_{loc}(\mathbb{R}^d)$, the inclusion $W^{1,p}(\mathbb{R}^d,\gamma_d)\subset\{f\in W^{1,p}_{loc}(\mathbb{R}^d):\ f,\ D_if\in L^p(\mathbb{R}^d,\gamma_d),\ i=1,\dots d\}$ is proved.

Conversely, let $f\in W^{1,p}_{loc}(\mathbb{R}^d)$ be such that $f,\ D_if\in L^p(\mathbb{R}^d,\gamma_d)$ for $i=1,\dots d$. Fix any function $\theta\in C^\infty_c(\mathbb{R}^d)$ such that $\theta\equiv 1$ in $B(0,1)$ and $\theta\equiv 0$ outside $B(0,2)$. For every $n\in\mathbb{N}$, we define

$$f_n(x):=\theta(x/n)f(x),\quad x\in\mathbb{R}^d.$$

Each f_n belongs to $W^{1,p}(\mathbb{R}^d,\gamma_d)$, because the restriction of f to $B(0,2n)$ may be approximated by a sequence (φ_k) of C^1 functions in $W^{1,p}(B(0,2n),\lambda_d)$, and the sequence (u_k) defined by $u_k(x)=\theta(x/n)\varphi_k(x)$ for $|x|\le 2n$, $u_k(x)=0$ for $|x|\ge 2n$ is contained in

$C_b^1(\mathbb{R}^d)$, it is a Cauchy sequence in the norm (9.1.5), and it converges to f_n in $L^p(\mathbb{R}^d, \gamma_d)$ since

$$\int_{\mathbb{R}^d} |u_k - f_n|^p d\gamma_d = \int_{B(0,2n)} |\theta(x/n)(f(x) - \varphi_k(x))|^p \gamma_d(dx) \leq \frac{\|\theta\|_\infty}{(2\pi)^{d/2}} \int_{B(0,2n)} |f - \varphi_k|^p dx.$$

In its turn, the sequence (f_n) converges to f in $L^p(\mathbb{R}^d, \gamma_d)$, by the Dominated Convergence Theorem. Moreover, for every $i = 1, \ldots, d$ we have $\partial f_n/\partial x_i(x) = n^{-1} D_i\theta(x/n)f(x) + \theta(x/n)D_i f(x)$, so that $\partial f_n/\partial x_i$ converges to $D_i f$ in $L^p(\mathbb{R}^d, \gamma_d)$, still by the Dominated Convergence Theorem. Therefore, $f \in W^{1,p}(\mathbb{R}^d, \gamma_d)$. □

By Proposition 9.1.6, if a C^1 function f is such that f, $D_i f$ belong to $L^p(\mathbb{R}^d, \gamma_d)$ for every $i = 1, \ldots, d$, then $f \in W^{1,p}(\mathbb{R}^d, \gamma_d)$. In particular, all polynomials belong to $W^{1,p}(\mathbb{R}^d, \gamma_d)$, for every $1 \leq p < +\infty$.

9.2 The Bochner integral

We only need the first notions of the theory of integration for Banach space valued functions. We refer to the books [DU], [Y, Ch. V] for a detailed treatment.

Let $(\Omega, \mathscr{F})$ be a measurable space and let $\mu : \mathscr{F} \to [0, +\infty)$ be a positive finite measure. We shall define integrals and L^p spaces of Y-valued functions, where Y is any separable real Banach space, with norm $\|\cdot\|_Y$.

In the following sections, Ω will be a Banach space X endowed with a Gaussian measure, and Y will be either X or the Cameron–Martin space H. However, the definitions and the basic properties are the same for a Gaussian measure and for a general positive finite measure.

As in the scalar valued case, the *simple functions* are functions of the type

$$F(x) = \sum_{i=1}^{n} 1\!\!1_{\Gamma_i}(x) y_i, \quad x \in \Omega,$$

with $n \in \mathbb{N}$, $\Gamma_i \in \mathscr{F}$, $y_i \in Y$ for every $i = 1, \ldots, n$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. In this case, the integral of F is defined by

$$\int_\Omega F(x)\mu(dx) := \sum_{i=1}^{n} \mu(\Gamma_i) y_i. \tag{9.2.1}$$

It is easily seen that the integral is linear, namely for every α, $\beta \in \mathbb{R}$ and for every couple of simple functions F_1, F_2

$$\int_\Omega (\alpha F_1(x) + \beta F_2(x))\mu(dx) = \alpha \int_\Omega F_1(x)\mu(dx) + \beta \int_X F_2(x)\mu(dx) \tag{9.2.2}$$

and it satisfies

$$\left\| \int_\Omega F(x)\gamma(dx) \right\|_Y \leq \int_\Omega \|F(x)\|_Y \gamma(dx), \tag{9.2.3}$$

for every simple function F (notice that $x \mapsto \|F(x)\|_Y$ is a simple real valued function).

Definition 9.2.1. *A function $F : \Omega \to Y$ is called strongly measurable if there exists a sequence of simple functions (F_n) such that $\lim_{n\to\infty} \|F(x) - F_n(x)\|_Y = 0$, for μ-a.e. $x \in \Omega$.*

Notice that if Y is separable then this notion coincides with the general notion of measurable function given in Definition 1.1.6, see [VTC, Proposition I.1.9]. If $Y = \mathbb{R}$, see Exercise 9.3. Also, notice that if F is strongly measurable, then $\|F(\cdot)\|_Y$ is a real valued measurable function. The following theorem is a consequence of an important result is due to Pettis (e.g. [DU, Thm. II.2]).

Theorem 9.2.2. *A function $F : \Omega \to Y$ is strongly measurable if and only if for every $f \in Y^*$ the composition $f \circ F : \Omega \to \mathbb{R}$, $x \mapsto f(F(x))$, is measurable.*

As a consequence, if Y is a separable Hilbert space and $\{y_k : k \in \mathbb{N}\}$ is an orthonormal basis of Y, then $F : \Omega \to Y$ is strongly measurable if and only if the real valued functions $x \mapsto \langle F(x), y_k\rangle_Y$ are measurable.

Definition 9.2.3. *A strongly measurable function $F : \Omega \to Y$ is called Bochner integrable if there exists a sequence of simple functions (F_n) such that*

$$\lim_{n\to\infty} \int_\Omega \|F(x) - F_n(x)\|_Y \mu(dx) = 0.$$

In this case, the sequence $\int_\Omega F_n d\mu$ is a Cauchy sequence in Y by estimate (9.2.3), and we define

$$\int_\Omega F(x)\,\mu(dx) := \lim_{n\to\infty} \int_\Omega F_n(x)\,\mu(dx)$$

(of course, the above limit is independent of the defining sequence (F_n)). The following result is known as the Bochner Theorem.

Proposition 9.2.4. *A measurable function $F : \Omega \to Y$ is Bochner integrable if and only if*

$$\int_\Omega \|F(x)\|_Y \mu(dx) < \infty.$$

Proof. If F is integrable, for every sequence of simple functions (F_n) in Definition 9.2.3 we have

$$\int_\Omega \|F(x)\|_Y \mu(dx) \le \int_\Omega \|F(x) - F_n(x)\|_Y \mu(dx) + \int_\Omega \|F_n(x)\|_Y \mu(dx),$$

which is finite for n large enough.

To prove the converse, if $\int_\Omega \|F(x)\|_Y \mu(dx) < \infty$ we construct a sequence of simple functions (F_n) that converge pointwise to F and such that $\lim_{n\to\infty} \int_\Omega \|F(x) - F_n(x)\|_Y \mu(dx) = 0$.

Let $\{y_k : k \in \mathbb{N}\}$ be a dense subset of Y. Set

$$\theta_n(x) := \min\{\|F(x) - y_k\|_Y : k = 1, \dots, n\},$$
$$k_n(x) := \min\{k \le n : \theta_n(x) = \|F(x) - y_k\|_Y\},$$

and

$$F_n(x) := y_{k_n(x)}, \quad x \in X.$$

Then every θ_n is a real valued measurable function. This implies that F_n is a simple function, because it takes the values $y_1, \dots y_n$, and for every $k = 1, \dots, n$, $F_n^{-1}(y_k)$ is the measurable set $\Gamma_k = \{x \in \Omega : \theta_n(x) = \|F(x) - y_k\|_Y\}$.

For every x the sequence $\|F_n(x) - F(x)\|_Y$ decreases monotonically to 0 as $n \to \infty$. Moreover, for every $n \in \mathbb{N}$,

$$\|F_n(x) - F(x)\|_Y \leq \|y_1 - F(x)\|_Y \leq \|y_1\|_Y + \|F(x)\|_Y, \quad x \in X. \tag{9.2.4}$$

By the Dominated Convergence Theorem (recall that μ is a finite measure) or else, by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_\Omega \|F_n(x) - F(x)\|_Y \mu(dx) = 0.$$

□

If $F : \Omega \to Y$ is integrable, for every $E \in \mathscr{F}$ the function $\mathbb{1}_E F$ is integrable, and we set

$$\int_E F(x)\mu(dx) = \int_\Omega \mathbb{1}_E(x)F(x)\,\mu(dx).$$

The Bochner integral is linear with respect to F, namely for every α, $\beta \in \mathbb{R}$ and for every integrable F_1, F_2, (9.2.2) holds. Moreover, it enjoys the following properties.

Proposition 9.2.5. *Let $F : \Omega \to Y$ be a Bochner integrable function. Then*

(i) $\| \int_\Omega F(x)\,\mu(dx)\|_Y \leq \int_\Omega \|F(x)\|_Y \mu(dx)$;

(ii) $\lim_{\mu(E)\to 0} \int_E F(x)\,\mu(dx) = 0$;

(iii) If (E_n) is a sequence of pairwise disjoint measurable sets in Ω and $E = \cup_{n\in\mathbb{N}} E_n$, then

$$\int_E F(x)\,\mu(dx) = \sum_{n\in\mathbb{N}} \int_{E_n} F(x)\,\mu(dx);$$

(iv) For every $f \in Y^*$, the real valued function $x \mapsto f(F(x))$ is in $L^1(\Omega,\mu)$, and

$$f\Big(\int_\Omega F(x)\,\mu(dx)\Big) = \int_\Omega f(F(x))\,\mu(dx). \tag{9.2.5}$$

Proof. (i) Let (F_n) be a sequence of simple functions as in Definition 9.2.3. By (9.2.3) for every $n \in \mathbb{N}$ we have $\| \int_\Omega F_n(x)\,\mu(dx)\|_Y \leq \int_\Omega \|F_n(x)\|_Y \mu(dx)$. Then,

$$\begin{aligned}\Big\| \int_\Omega F(x)\,\mu(dx)\Big\|_Y &= \Big\| \lim_{n\to\infty} \int_\Omega F_n(x)\,\mu(dx)\Big\|_Y \leq \limsup_{n\to\infty} \int_\Omega \|F_n(x)\|_Y\,\mu(dx) \\ &\leq \lim_{n\to\infty} \int_\Omega \|F_n(x) - F(x)\|_Y\,\mu(dx) + \int_\Omega \|F(x)\|_Y\,\mu(dx) \\ &= \int_\Omega \|F(x)\|_Y\,\mu(dx).\end{aligned}$$

Statement (ii) means: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) \le \delta$ implies $\| \int_E F(x)\,\mu(dx)\|_Y \le \varepsilon$. Since $\lim_{\mu(E)\to 0} \int_E \|F(x)\|_Y\,\mu(dx) = 0$, statement (ii) is a consequence of (i).

Let us prove statement (iii). Since, for every n,

$$\left\| \int_{E_n} F(x)\,\mu(dx) \right\|_Y \le \int_{E_n} \|F(x)\|_Y \mu(dx),$$

the series $\sum_{n\in\mathbb{N}} \int_{E_n} F(x)\,\mu(dx)$ converges in Y, and its norm does not exceed

$$\int_\Omega \|F(x)\|_Y \mu(dx).$$

Since the Bochner integral is finitely additive,

$$\left\| \int_E F(x)\,\mu(dx) - \sum_{n=1}^m \int_{E_n} F(x)\,\mu(dx) \right\|_Y = \left\| \int_{\cup_{n=m+1}^\infty E_n} F(x)\,\mu(dx) \right\|_Y$$

where $\lim_{m\to\infty} \mu(\cup_{n=m+1}^\infty E_n) = 0$. By statement (ii), the right-hand side vanishes as $m \to \infty$, and statement (iii) follows.

Let us prove statement (iv). Note that (9.2.5) holds obviously for simple functions. Let (F_n) be the sequence of functions in the proof of Proposition 9.2.4. Then,

$$\begin{aligned} f\Big(\int_\Omega F(x)\,\mu(dx) \Big) &= f\Big(\lim_{n\to\infty} \int_\Omega F_n(x)\,\mu(dx) \Big) \\ &= \lim_{n\to\infty} f\Big(\int_\Omega F_n(x)\,\mu(dx) \Big) = \lim_{n\to\infty} \int_\Omega f(F_n(x))\,\mu(dx). \end{aligned}$$

On the other hand, the sequence $(f(F_n(x)))$ converges pointwise to $f(F(x))$, and by (9.2.4)

$$\begin{aligned} |f(F_n(x))| &\le \|f\|_{Y^*} \|F_n(x)\|_Y \le \|f\|_{Y^*} (\|F_n(x) - F(x)\|_Y + \|F(x)\|_Y) \\ &\le \|f\|_{Y^*} (\|y_1\|_Y + 2\|F(x)\|_Y). \end{aligned}$$

By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_\Omega f(F_n(x))\,\mu(dx) = \int_\Omega f(F(x))\,\mu(dx),$$

and the statement follows. □

Remark 9.2.6. As a consequence of (iv), if Y is a separable Hilbert space and $\{y_k :\ k \in \mathbb{N}\}$ is an orthonormal basis of Y, for every Bochner integrable $F : \Omega \to Y$ the real valued functions $x \mapsto \langle F(x), y_k\rangle_Y$ belong to $L^1(\Omega,\mu)$, and we have

$$\int_\Omega F(x)\mu(dx) = \sum_{k=1}^\infty \int_\Omega \langle F(x), y_k\rangle_Y \mu(dx)\, y_k.$$

The L^p spaces of Y-valued functions are defined as in the scalar case. Namely, for every $1 \le p < +\infty$, $L^p(\Omega, \mu; Y)$ is the space of the (equivalence classes) of Bochner integrable functions $F : \Omega \to Y$ such that

$$\|F\|_{L^p(\Omega,\mu;Y)} := \left(\int_X \|F(x)\|_Y^p \mu(dx) \right)^{1/p} < \infty.$$

The proof that $L^p(\Omega, \mu; Y)$ is a Banach space with the above norm is the same as in the real valued case. If $p = 2$ and Y is a Hilbert space, $L^p(\Omega, \mu; Y)$ is a Hilbert space with the scalar product

$$\langle F, G\rangle_{L^2(\Omega,\mu;Y)} := \int_\Omega \langle F(x), G(x)\rangle_Y \mu(dx).$$

As usual, we define

$$L^\infty(\Omega, \mu; Y) := \Big\{ F : \Omega \to Y \text{ measurable s.t. } \|F\|_{L^\infty(\Omega,\mu;Y)} < +\infty \Big\},$$

where

$$\|F\|_{L^\infty(\Omega,\mu;Y)} := \inf\Big\{ M > 0 : \mu(\{x : \|F(x)\|_Y > M\}) = 0 \Big\}.$$

Notice that if Y is a separable Hilbert space, which is our setting, the space $L^p(\Omega, \mu; Y)$ is reflexive for $1 < p < \infty$, see [DU, Section IV.1].

The first example of Bochner integral that we met in these lectures was the mean a of a Gaussian measure γ on a separable Banach space X. By Proposition 2.3.3, there exists a unique $a \in X$ such that $a_\gamma(f) = f(a)$, for every $f \in X^*$. Since γ is a Borel measure, every continuous $F : X \to X$ is measurable; in particular $F(x) := x$ is measurable, hence strongly measurable. By the Fernique Theorem and Proposition 9.2.4 it belongs to $L^p(X, \gamma; X)$ for every $1 \le p < +\infty$, and we have

$$a = \int_X x\, \gamma(dx).$$

Indeed, for every $f \in X^*$, we have

$$f\left(\int_X x\, \gamma(dx) \right) = \int_X f(x)\, \gamma(dx) = a_\gamma(f),$$

by (9.2.5). Therefore, $a = \int_X x\, \gamma(dx)$.

9.3 The infinite dimensional case

9.3.1 Differentiable functions

Definition 9.3.1. *Let X, Y be normed spaces. Let $\overline{x} \in X$ and let Ω be a neighbourhood of $\overline{x}$. A function $f : \Omega \to Y$ is called (Fréchet) differentiable at $\overline{x}$ if there exists $\ell \in \mathcal{L}(X, Y)$ such that*

$$\|f(\overline{x} + h) - f(\overline{x}) - \ell(h)\|_Y = o(\|h\|_X) \quad \text{as } h \to 0 \text{ in } X.$$

In this case, ℓ is unique, and we set $f'(\overline{x}) := \ell$.

Several properties of differentiable functions may be proved as in the case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$. First, if f is differentiable at $\overline{x}$ it is continuous at $\overline{x}$. Moreover, for every $v \in X$ the directional derivative

$$\frac{\partial f}{\partial v}(\overline{x}) := Y - \lim_{t \to 0} \frac{f(\overline{x} + tv) - f(\overline{x})}{t}$$

exists and is equal to $f'(\overline{x})(v)$.

If $Y = \mathbb{R}$ and $f : X \to \mathbb{R}$ is differentiable at $\overline{x}$, $f'(\overline{x})$ is an element of X^*. In particular, if $f \in X^*$ then f is differentiable at every $\overline{x}$ and f' is constant, with $f'(\overline{x})(v) = f(v)$ for every $\overline{x}$, $v \in X$. If $f \in \mathcal{F}C^1_b(X)$, $f(x) = \varphi(\ell_1(x), \ldots, \ell_n(x))$ with $\ell_k \in X^*$, $\varphi \in C^1_b(\mathbb{R}^n)$, f is differentiable at every $\overline{x}$ and

$$f'(\overline{x})(v) = \sum_{k=1}^{n} \frac{\partial \varphi}{\partial \xi_k}((\ell_1(\overline{x}), \ldots, \ell_n(\overline{x}))\ell_k(v), \quad \overline{x},\ v \in X.$$

If f is differentiable at x for every x in a neighbourhood of $\overline{x}$, it may happen that the function $X \to \mathcal{L}(X, Y)$, $x \mapsto f'(x)$ is differentiable at $\overline{x}$, too. In this case, the derivative is denoted by $f''(\overline{x})$, and it is an element of $\mathcal{L}(X, \mathcal{L}(X, Y))$. The higher order derivatives are defined recursively, in the same way.

If $f : X \to \mathbb{R}$ is twice differentiable at $\overline{x}$, $f''(\overline{x})$ is an element of $\mathcal{L}(X, X^*)$, which is canonically identified with the space of the continuous bilinear forms $\mathcal{L}^{(2)}(X)$: indeed, if $v \in \mathcal{L}(X, X^*)$, the function $X^2 \to \mathbb{R}$, $(x, y) \mapsto v(x)(y)$, is linear both with respect to x and with respect to y and it is continuous, so that it is a bilinear form; conversely every bilinear continuous form $a : X^2 \to \mathbb{R}$ gives rise to the element $v \in \mathcal{L}(X, X^*)$ defined by $v(x)(y) = a(x, y)$. Moreover,

$$\|v\|_{\mathcal{L}(X,X^*)} = \sup_{x \neq 0,\, y \neq 0} \frac{|v(x)(y)|}{\|x\|_X \, \|y\|_X} = \sup_{x \neq 0,\, y \neq 0} \frac{|a(x, y)|}{\|x\|_X \, \|y\|_X} = \|a\|_{\mathcal{L}^{(2)}(X)}.$$

Similarly, if $f : X \to \mathbb{R}$ is k times differentiable at $\overline{x}$, $f^{(k)}(\overline{x})$ is identified with an element of the space $\mathcal{L}^{(k)}(X)$ of the continuous k-linear forms.

Definition 9.3.2. *Let $k \in \mathbb{N}$. We denote by $C^k_b(X)$ the set of bounded and k times continuously differentiable functions $f : X \to \mathbb{R}$, with bounded $\|f^{(j)}\|_{\mathcal{L}^{(j)}(X)}$ for every $j = 1, \ldots, k$. It is normed by*

$$\|f\|_{C^k_b(X)} = \sum_{j=0}^{k} \sup_{x \in X} \|f^{(j)}(x)\|_{\mathcal{L}^{(j)}(X)},$$

where we set $f^{(0)}(x) = f(x)$. Moreover we set

$$C^\infty_b(X) = \bigcap_{k \in \mathbb{N}} C^k_b(X).$$

If X is a Hilbert space and $f : X \to \mathbb{R}$ is differentiable at $\overline{x}$, there exists a unique $y \in X$ such that $f'(\overline{x})(x) = \langle x, y\rangle$, for every $x \in X$. We set

$$\nabla f(\overline{x}) := y.$$

From now on, X is a separable Banach space endowed with a norm $\|\cdot\|$ and with a Gaussian centred non degenerate measure γ, and H is its Cameron-Martin space defined in Lecture 3.

Definition 9.3.3. *A function $f : X \to \mathbb{R}$ is called H-differentiable at $\overline{x} \in X$ if there exists $\ell_0 \in H^*$ such that*

$$|f(\overline{x}+h) - f(\overline{x}) - \ell_0(h)| = o(|h|_H) \quad \text{as } h \to 0 \text{ in } H.$$

If f is H-differentiable at $\overline{x}$, the operator ℓ_0 in the definition is called H-derivative of f at $\overline{x}$, and there exists a unique $y \in H$ such that $\ell_0(h) = [h, y]_H$ for every $h \in H$. We set

$$\nabla_H f(\overline{x}) := y.$$

Definition 9.3.3 differs from 9.3.1 in that the increments are taken only in H.

Lemma 9.3.4. *If f is differentiable at $\overline{x}$, then it is H-differentiable at $\overline{x}$, with H-derivative given by $h \mapsto f'(\overline{x})(h)$ for every $h \in H$. Moreover, we have*

$$\nabla_H f(\overline{x}) = R_\gamma f'(\overline{x}). \tag{9.3.1}$$

Proof. Setting $\ell = f'(\overline{x})$ we have

$$\lim_{|h|_H \to 0} \frac{|f(\overline{x}+h) - f(\overline{x}) - \ell(h)|}{|h|_H} = \lim_{|h|_H \to 0} \frac{|f(\overline{x}+h) - f(\overline{x}) - \ell(h)|}{\|h\|} \frac{\|h\|}{|h|_H} = 0,$$

because H is continuously embedded in X so that the ratio $\|h\|/|h|_H$ is bounded by a constant independent of h. This proves the first assertion. To prove (9.3.1), we recall that for every $\varphi \in X^*$ we have $\varphi(h) = [R_\gamma \varphi, h]_H$ for each $h \in H$; in particular, taking $\varphi = f'(\overline{x})$ we obtain $f'(\overline{x})(h) = [R_\gamma f'(\overline{x}), h]_H = [\nabla_H f(\overline{x}), h]_H$ for each $h \in H$, and therefore $\nabla_H f(\overline{x}) = R_\gamma f'(\overline{x})$. □

If f is just H-differentiable at $\overline{x}$, the directional derivative $\frac{\partial f}{\partial v}(\overline{x})$ exists for every $v \in H$, and it is given by $[\nabla_H f(\overline{x}), v]_H$. Fixed any orthonormal basis $\{h_n : n \in \mathbb{N}\}$ of H, we set

$$\partial_i f(\overline{x}) := \frac{\partial f}{\partial h_i}(\overline{x}), \quad i \in \mathbb{N}.$$

So, we have

$$\nabla_H f(\overline{x}) = \sum_{i=1}^{\infty} \partial_i f(\overline{x}) h_i, \tag{9.3.2}$$

where the series converges in H.

We warn the reader that if X is a Hilbert space and f is differentiable at $\overline{x}$, the gradient and the H-gradient of f at $\overline{x}$ do not coincide in general. If $\gamma = \mathscr{N}(0,Q)$, identifying X^* with X as usual, Lemma 9.3.4 implies that $\nabla_H f(\overline{x}) = Q\nabla f(\overline{x})$.

We recall that if γ is non degenerate, then Q is positive definite. Fixed any orthonormal basis $\{e_j :\ j\in\mathbb{N}\}$ of X consisting of eigenvectors of Q, $Qe_j = \lambda_j e_j$, then a canonical orthonormal basis of H is $\{h_j :\ j\in\mathbb{N}\}$, with $h_j = \sqrt{\lambda_j}e_j$, and we have

$$\partial_j f(\overline{x}) = \sqrt{\lambda_j}\frac{\partial f}{\partial e_j}(\overline{x}), \quad j\in\mathbb{N}.$$

9.3.2 Sobolev spaces of order 1

As in finite dimension, the starting point to define the Sobolev spaces is an integration formula for C^1_b functions.

Proposition 9.3.5. *For every $f\in C^1_b(X)$ and $h\in H$ we have*

$$\int_X \frac{\partial f}{\partial h}\,d\gamma = \int_X f\,\hat{h}\,d\gamma. \tag{9.3.3}$$

Consequently, for every f, $g\in C^1_b(X)$ and $h\in H$ we have

$$\int_X \frac{\partial f}{\partial h} g\,d\gamma = -\int_X \frac{\partial g}{\partial h} f\,d\gamma + \int_X f\,g\,\hat{h}\,d\gamma. \tag{9.3.4}$$

Proof. By the Cameron–Martin Theorem 3.1.5, for every $t\in\mathbb{R}$ we have

$$\int_X f(x+th)\,\gamma(dx) = \int_X f(x)e^{t\hat{h}(x)-t^2|h|^2_H/2}\gamma(dx),$$

so that, for $0<|t|\leq 1$,

$$\int_X \frac{f(x+th)-f(x)}{t}\,\gamma(dx) = \int_X f(x)\,\frac{e^{t\hat{h}(x)-t^2|h|^2_H/2}-1}{t}\,\gamma(dx).$$

As $t\to 0$, the integral in the left-hand side converges to $\int_X \partial f/\partial h\,d\gamma$, by the Dominated Convergence Theorem. Concerning the right-hand side, $(e^{t\hat{h}(x)-t^2|h|^2_H/2}-1)/t \to \hat{h}(x)$ for every $x\in X$. We estimate

$$\begin{aligned}\left|\frac{e^{t\hat{h}(x)-t^2|h|^2_H/2}-1}{t}\right| &= \left|\frac{e^{-t^2|h|^2_H/2}(e^{t\hat{h}(x)}-1)}{t} + \frac{e^{-t^2|h|^2_H/2}-1}{t}\right| \\ &\leq |\hat{h}(x)e^{t\hat{h}(x)}| + \sup_{0<t\leq 1}\left|\frac{e^{-t^2|h|^2_H/2}-1}{t}\right|,\end{aligned}$$

where the function $x\mapsto \hat{h}(x)e^{t\hat{h}(x)}$ belongs to $L^1(X,\gamma)$ since $\hat{h}$ is a Gaussian random variable. So, applying the Dominated Convergence Theorem we get the statement. $\square$

Notice that formula (9.3.3) is a natural extension of (9.1.2) to the infinite dimensional case. In $(\mathbb{R}^d, \gamma_d)$ the equality $H = \mathbb{R}^d$ holds, and for every $h \in \mathbb{R}^d$ we have $\hat{h}(x) = h \cdot x = [h, x]_H$.

We proceed as in finite dimension to define the Sobolev spaces of order 1. Next step is to prove that some gradient operator, defined on a set of good enough functions, is closable in $L^p(X, \gamma)$. In our general setting the only available gradient is ∇_H. We shall use the following lemma, whose proof is left as an exercise, being a consequence of the results of Lecture 7.

Lemma 9.3.6. *Let $\psi \in L^1(X, \gamma)$ be such that*

$$\int_X \psi\, \varphi\, d\gamma = 0, \quad \varphi \in \mathcal{F}C_b^1(X).$$

Then $\psi = 0$ a.e.

Proposition 9.3.7. *For every $1 \le p < +\infty$, the operator $\nabla_H : D(\nabla_H) = \mathcal{F}C_b^1(X) \to L^p(X, \gamma; H)$ is closable in $L^p(X, \gamma)$.*

Proof. Let $1 < p < +\infty$. Let $f_n \in \mathcal{F}C_b^1(X)$ be such that $f_n \to 0$ in $L^p(X, \gamma)$ and $\nabla_H f_n \to G$ in $L^p(X, \gamma; H)$. For every $h \in H$ and $\varphi \in \mathcal{F}C_b^1(X)$ we have

$$\lim_{n\to\infty} \int_X \frac{\partial f_n}{\partial h} \varphi\, d\gamma = \int_X [G(x), h]_H \varphi(x)\, \gamma(dx),$$

since

$$\int_X |(\partial f_n/\partial h - [G(x), h]_H)\varphi|\, d\gamma \le |h|_H^p \left(\int_X |\nabla_H f_n - G|_H^p d\gamma \right)^{1/p} \|\varphi\|_{L^{p'}(X,\gamma)}.$$

On the other hand,

$$\int_X \frac{\partial f_n}{\partial h} \varphi\, d\gamma = -\int_X f_n \frac{\partial \varphi}{\partial h}\, d\gamma + \int_X f_n \varphi \hat{h}\, d\gamma, \quad n \in \mathbb{N},$$

so that, since $f_n \to 0$ in $L^p(X, \gamma)$ and $\partial\varphi/\partial h$, $\hat{h}\varphi \in L^{p'}(X, \gamma)$,

$$\lim_{n\to\infty} \int_X \frac{\partial f_n}{\partial h} \varphi\, d\gamma = 0.$$

So,

$$\int_X [G(x), h]_H \varphi(x)\, \gamma(dx) = 0, \quad \varphi \in \mathcal{F}C_b^1(X), \tag{9.3.5}$$

and by Lemma 9.3.6, $[G(x), h]_H = 0$ a.e. Fix any orthonormal basis $\{h_k : k \in \mathbb{N}\}$ of H. Then $\cup_{k\in\mathbb{N}}\{x : [G(x), h_k]_H \ne 0\}$ is negligible so that $G(x) = 0$ a.e.

Let now be $p = 1$. The above procedure does not work, since $\hat{h}\varphi \notin L^\infty(X, \gamma)$ in general, although it belongs to $L^q(X, \gamma)$ for every $q > 1$. We modify it introducing a function $\theta \in C_b^2(\mathbb{R})$ such that $\theta(0) = 0$, $\theta'(0) \ne 0$, and replacing f_n by $\theta \circ f_n$. Still,

$\theta \circ f_n \to 0$ in $L^1(X,\gamma)$ as $n \to \infty$, because $|\theta(f_n(x))| = |\theta(f_n(x)) - \theta(0)| \le \|\theta'\|_\infty |f_n(x)|$. Moreover, for every $h \in H$, $n \in \mathbb{N}$ and $\varphi \in \mathcal{F}C_b^1(X)$ we have

$$\int_X \frac{\partial(\theta \circ f_n)}{\partial h} \varphi \, d\gamma = \int_X (\theta' \circ f_n) \frac{\partial f_n}{\partial h} \varphi \, d\gamma. \tag{9.3.6}$$

Letting $n \to \infty$, the right hand side converges to $\theta'(0) \int_X [G(x), h]_H \varphi \, d\gamma$. Indeed,

$$\begin{aligned}
&\left| \int_X (\theta' \circ f_n) \frac{\partial f_n}{\partial h} \varphi \, d\gamma - \theta'(0) \int_X [G(x), h]_H \varphi \, d\gamma \right| \\
&\le \int_X |(\theta' \circ f_n)\varphi| \left| \frac{\partial f_n}{\partial h} - [G(x), h]_H \right| d\gamma + \int_X |(\theta' \circ f_n) - \theta'(0)| |[G(x), h]_H \varphi| \, d\gamma.
\end{aligned}$$

Since θ' and φ are bounded, the first integral vanishes as $n \to \infty$. Every subsequence of (f_n) has a sub-subsequence $f_{n_k} \to 0$ a.e., so that $(\theta' \circ f_{n_k}) - \theta'(0) \to 0$ a.e.; moreover,

$$|(\theta' \circ f_{n_k})(x) - \theta'(0)| |[G(x), h]_H \varphi(x)| \le 2\|\theta'\|_\infty |G(x)|_H |h|_H \|\varphi\|_\infty \le C|G(x)|_H$$

and by the Dominated Convergence theorem $\int_X |(\theta' \circ f_{n_k}) - \theta'(0)| |[G(x), h]_H \varphi| \, d\gamma \to 0$ as $k \to \infty$. Since this holds for *every* subsequence of (f_n), then

$$\lim_{n \to \infty} \int_X |(\theta' \circ f_n) - \theta'(0)| |[G(x), h]_H \varphi| \, d\gamma = 0.$$

Similar arguments yield

$$\lim_{n \to \infty} \int_X \frac{\partial(\theta \circ f_n)}{\partial h} \varphi \, d\gamma = 0. \tag{9.3.7}$$

Indeed, by the integration by parts formula (9.3.4), for every n we have

$$\int_X \frac{\partial(\theta \circ f_n)}{\partial h} \varphi \, d\gamma = \int_X (\theta \circ f_n)[\hat{h}\varphi - \frac{\partial \varphi}{\partial h}] \, d\gamma.$$

Taking as before any subsequence (f_{n_k}) such that $f_{n_k} \to 0$ a.e., we have $(\theta \circ f_{n_k})[\hat{h}\varphi - \frac{\partial \varphi}{\partial h}] \to 0$ a.e., and for a.e. $x \in X$ we have

$$\left| (\theta \circ f_{n_k}) \Big[\hat{h}\varphi - \frac{\partial \varphi}{\partial h} \Big](x) \right| \le \|\theta\|_\infty \left(|\hat{h}(x)| \|\varphi\|_\infty + \left\| \frac{\partial \varphi}{\partial h} \right\|_\infty \right)$$

and by the Dominated Convergence Theorem, (9.3.7) holds along the subsequence (f_{n_k}) and then along the whole sequence (f_n).

Letting $n \to \infty$ in (9.3.6), we obtain

$$\theta'(0) \int_X [G(x), h]_H \varphi \, d\gamma = 0, \quad \varphi \in \mathcal{F}C_b^1(X).$$

Since $\theta'(0) \ne 0$, (9.3.5) holds for every $\varphi \in \mathcal{F}C_b^1(X)$. By Lemma 9.3.6, $[G(x), h]_H = 0$ a.e. for every $h \in H$ and we conclude as in the case $1 < p < +\infty$. □

The proof of Proposition 9.3.7 for $p = 1$ is more complicated than the proof of Lemma 9.1.3, where we could use compactly supported functions φ.

Remark 9.3.8. Note that in the proof of Proposition 9.3.7 we proved that for every $h \in H$ the linear operator $\partial_h : D(\partial_h) = \mathcal{F}C_b^1(X) \to L^p(X, \gamma)$ is closable in $L^p(X, \gamma)$.

We are now ready to define the Sobolev spaces of order 1 and the generalized H-gradients.

Definition 9.3.9. *For every $1 \leq p < +\infty$, $W^{1,p}(X, \gamma)$ is the domain of the closure of $\nabla_H : \mathcal{F}C_b^1(X) \to L^p(X, \gamma; H)$ in $L^p(X, \gamma)$ (still denoted by ∇_H). Therefore, an element $f \in L^p(X, \gamma)$ belongs to $W^{1,p}(X, \gamma)$ iff there exists a sequence of functions $f_n \in \mathcal{F}C_b^1(X)$ such that $f_n \to f$ in $L^p(X, \gamma)$ and $\nabla_H f_n$ converges in $L^p(X, \gamma; H)$, and in this case, $\nabla_H f = \lim_{n\to\infty} \nabla_H f_n$.*

$W^{1,p}(X, \gamma)$ is a Banach space with the graph norm

$$\|f\|_{W^{1,p}} := \|f\|_{L^p(X,\gamma)} + \|\nabla_H f\|_{L^p(X,\gamma;H)} = \left(\int_X |f|^p d\gamma\right)^{1/p} + \left(\int_X |\nabla_H f|_H^p d\gamma\right)^{1/p}. \tag{9.3.8}$$

For $p = 2$, $W^{1,2}(X, \gamma)$ is a Hilbert space with the natural inner product

$$\langle f, g\rangle_{W^{1,2}} := \int_X f\, g\, d\gamma + \int_X [\nabla_H f, \nabla_H g]_H d\gamma,$$

which gives an equivalent norm.

For every fixed orthonormal basis $\{h_j : \ j \in \mathbb{N}\}$ of H, and for every $f \in W^{1,p}(X, \gamma)$, we set

$$\partial_j f(x) := [\nabla_H f(x), h_j]_H, \quad j \in \mathbb{N}.$$

More generally, for every $h \in H$ we set

$$\partial_h f(x) := [\nabla_H f(x), h]_H.$$

By definition,

$$\int_X |\nabla_H f|_H^p d\gamma = \int_X \left(\sum_{j=1}^\infty [\nabla_H f, h_j]_H^2\right)^{p/2} d\gamma = \int_X \left(\sum_{j=1}^\infty (\partial_j f)^2\right)^{p/2} d\gamma.$$

Moreover, if $f_n \in \mathcal{F}C_b^1(X)$ is such that $f_n \to f$ in $L^p(X, \gamma)$ and $\nabla_H f_n$ converges in $L^p(X, \gamma; H)$, then

$$\lim_{n\to\infty} [\nabla_H f_n, h_j]_H = \lim_{n\to\infty} \partial_j f_n = \partial_j f, \quad \text{in} \quad L^p(X, \gamma).$$

As in finite dimension, several properties of the spaces $W^{1,p}(X, \gamma)$ follow easily.

Proposition 9.3.10. *Let $1 < p < \infty$. Then*

(i) *the integration formula* (9.3.3) *holds for every* $f \in W^{1,p}(X,\gamma)$, $h \in H$;

(ii) *if* $\theta \in C^1_b(X;\mathbb{R})$ *and* $f \in W^{1,p}(X,\gamma)$, *then* $\theta \circ f \in W^{1,p}(X,\gamma)$, *and* $\nabla_H(\theta \circ f) = (\theta' \circ f)\nabla_H f$;

(iii) *if* $f \in W^{1,p}(X,\gamma)$, $g \in W^{1,q}(X,\gamma)$ *with* $1/p + 1/q = 1/s \le 1$, *then* $fg \in W^{1,s}(X,\gamma)$ *and*

$$\nabla_H(fg) = \nabla_H f\, g + f\nabla_H g;$$

(iv) $W^{1,p}(X,\gamma)$ *is reflexive;*

(v) *if* $f_n \to f$ *in* $L^p(X,\gamma)$ *and* $\sup_{n\in\mathbb{N}} \|f_n\|_{W^{1,p}(X,\gamma)} < \infty$, *then* $f \in W^{1,p}(X,\gamma)$.

Proof. The proof is just a rephrasing of the proof of Proposition 9.1.5.

Statement (i) follows approaching f by a sequence of functions belonging to $\mathcal{F}C^1_b(X)$, using (9.3.3) for every approximating function f_n and letting $n \to \infty$.

Statement (ii) follows approaching $\theta\circ f$ by $\theta\circ f_n$, if $(f_n) \subset \mathcal{F}C^1_b(X)$ is such that $f_n \to f$ in $L^p(X,\gamma)$ and $\nabla_H f_n \to \nabla_H f$ in $L^p(X,\gamma;H)$.

Statement (iii) follows from the definition, approaching fg by $f_n g_n$ if (f_n), $(g_n) \subset \mathcal{F}C^1_b(X)$, are such that $f_n \to f$ in $L^p(X,\gamma)$, $\nabla_H f_n \to \nabla_H f$ in $L^p(X,\gamma;H)$, $g_n \to g$ in $L^{p'}(X,\gamma)$, $\nabla_H g_n \to \nabla_H g$ in $L^{p'}(X,\gamma;H)$. Then $\lim_{n\to\infty} f_n g_n = fg$ in $L^s(X,\gamma)$, and the sequence $(\nabla_H(f_n g_n))$ converges to $g\nabla_H f + f\nabla_H g$ in $L^s(X,\gamma;H)$.

Let us prove (iv). The mapping $u \mapsto Tu = (u, \nabla_H u)$ is an isometry from $W^{1,p}(X,\gamma)$ to the product space $E := L^p(X,\gamma) \times L^p(X,\gamma;H)$, which implies that the range of T is closed in E. Now, $L^p(X,\gamma)$ and $L^p(X,\gamma;H)$ are reflexive (e.g. [DU, Ch. IV]) so that E is reflexive, and $T(W^{1,p}(X,\gamma))$ is reflexive too. Being isometric to a reflexive space, $W^{1,p}(X,\gamma)$ is reflexive.

As a consequence of reflexivity, if a sequence (f_n) is bounded in $W^{1,p}(X,\gamma)$ a subsequence f_{n_k} converges weakly to an element g of $W^{1,p}(X,\gamma)$ as $k \to \infty$. Since $f_{n_k} \to f$ in $L^p(X,\gamma)$, then $f = g$ and statement (v) is proved. □

As in finite dimension, statement (ii) holds as well for $p = 1$.

Remark 9.3.11. Let X be a Hilbert space and let $\gamma = \mathscr{N}(0,Q)$ with $Q > 0$. For every $f \in \mathcal{F}C^1_b(X)$ we have $\nabla_H f(x) = Q\nabla f(x)$, so that

$$|\nabla_H f(x)|_H^2 = \langle Q^{-1/2}Q\nabla f(x), Q^{-1/2}Q\nabla f(x)\rangle = \|Q^{1/2}\nabla f(x)\|^2,$$

and

$$\|f\|_{W^{1,p}(X,\gamma)} = \|f\|_{L^p(X,\gamma)} + \left(\int_X \|Q^{1/2}\nabla f(x)\|^p d\gamma\right)^{1/p}.$$

Fixed any orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of X consisting of eigenvectors of Q, $Qe_j = \lambda_j e_j$, then a canonical basis of H is $\{h_j : j \in \mathbb{N}\}$, with $h_j = \sqrt{\lambda_j}e_j$, $\partial_j f(x) = \sqrt{\lambda_j}\partial f/\partial e_j$, and

$$\|f\|_{W^{1,p}(X,\gamma)} = \|f\|_{L^p(X,\gamma)} + \left(\int_X \left(\sum_{j=1}^\infty \lambda_j \left(\frac{\partial f}{\partial e_j}\right)^2\right)^{1/p} d\gamma\right)^{1/p}.$$

One can consider Sobolev spaces $\widetilde{W}^{1,p}(X,\gamma)$ defined as in Definition 9.3.9, with the gradient ∇ replacing the H-gradient ∇_H. Namely, the proof of Proposition 9.3.7 yields that the operator $\nabla : \mathcal{F}C_b^1(X) \to L^p(X,\gamma;X)$ is closable; we define $\widetilde{W}^{1,p}(X,\gamma)$ as the domain of its closure, still denoted by ∇. This choice looks even simpler and more natural; the norm in $\widetilde{W}^{1,p}$ is the graph norm of ∇ and it is given by

$$\begin{aligned} \|f\|_{\widetilde{W}^{1,p}(X,\gamma)} &:= \left(\int_X |f|^p d\gamma\right)^{1/p} + \left(\int_X \|\nabla f\|^p d\gamma\right)^{1/p} \\ &= \left(\int_X |f|^p d\gamma\right)^{1/p} + \left(\int_X \left(\sum_{j=1}^{\infty} \left(\frac{\partial f}{\partial e_j}\right)^2\right)^{1/p} d\gamma\right)^{1/p}. \end{aligned} \tag{9.3.9}$$

Since $\lim_{k\to\infty} \lambda_k = 0$, our Sobolev space $W^{1,p}(X,\gamma)$ strictly contains $\widetilde{W}^{1,p}(X,\gamma)$, and the embedding $\widetilde{W}^{1,p}(X,\gamma) \subset W^{1,p}(X,\gamma)$ is continuous.

9.4 Exercises

Exercise 9.1. Prove Lemma 9.1.1.

Exercise 9.2. (i) Prove that statement (v) of Proposition 9.1.5 is false for $p = 1$, $d = 1$. (Hint: use Proposition 9.1.6, and the sequence (f_n) defined by $f_n(x) = 0$ for $x \le 0$, $f_n(x) = nx$ for $0 \le x \le 1/n$, $f_n(x) = 1$ for $x \ge 1/n$).

(ii) Using (i), prove that $W^{1,1}(\mathbb{R},\gamma_1)$ is not reflexive.

Exercise 9.3. Let $(\Omega, \mathscr{F})$ be a measurable space, and let μ be a positive finite measure in Ω. Prove that a function $f : \Omega \to \mathbb{R}$ is measurable if and only if it is the pointwise a.e. limit of a sequence of simple functions.

Exercise 9.4. Prove Lemma 9.3.6.

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