Lecture 8

Zero-One law and Wiener chaos

In this Lecture we introduce the Hermite polynomials, which provide an orthonormal basis in $L^2(X,\gamma)$. Accordingly, $L^2(X,\gamma)$ is decomposed as the Hilbert sum of the (mutually orthogonal) subspaces $\mathfrak{X}_k$ generated by the polynomials of degree $k \in \mathbb{N}$, see Proposition 8.1.9. Knowing explicitly an orthonormal basis in this not elementary setting is a real luxury! The term *chaos* has been introduced by Wiener in [W] and the structure that we discuss here is usually called *Wiener chaos*. Of course, the Hermite polynomials are used in several proofs, including that of the zero-one law. The expression "zero-one law" is used in different probabilistic contexts, where the final statement is that a certain event has probability either 0 or 1. In our case we show that every measurable subspace has measure either 0 or 1.

We work as usual in a separable Banach space X endowed with a centred Gaussian measure γ. The symbols R_γ, X^*_γ, H have the usual meaning.

8.1 Hermite polynomials

As first step, we introduce the Hermite polynomials and we present their main properties. We shall encounter them in many occasions; further properties will be presented when needed.

8.1.1 Hermite polynomials in finite dimension

To start with, we introduce the one dimensional Hermite polynomials.

Definition 8.1.1. *The sequence of Hermite polynomials in $\mathbb{R}$ is defined by*

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} \exp\{x^2/2\} \frac{d^k}{dx^k} \exp\{-x^2/2\}, \quad k \in \mathbb{N} \cup \{0\},\ x \in \mathbb{R}. \tag{8.1.1}$$

Then, $H_0(x) \equiv 1$, $H_1(x) = x$, $H_2(x) = (x^2 - 1)/\sqrt{2}$, $H_3(x) = (x^3 - 3x)/\sqrt{6}$, etc. Some properties of Hermite polynomials are listed below. Their proofs are easy, and left as exercises, see Exercise 8.1.

Lemma 8.1.2. *For every $k \in \mathbb{N}$, H_k is a polynomial of degree k, with positive leading coefficient. Moreover, for every $x \in \mathbb{R}$,*

$$\begin{cases} (i) \quad H_k'(x) = \sqrt{k} H_{k-1}(x) = x H_k(x) - \sqrt{k+1} H_{k+1}(x), \\ \\ (ii) \quad H_k''(x) - x H_k'(x) = -k H_k(x). \end{cases} \tag{8.1.2}$$

Note that formula (ii) says that H_k is an eigenfunction of the one dimensional *Ornstein–Uhlenbeck operator* $D^2 - xD$, with eigenvalue $-k$. This operator will play an important role in the next lectures.

Proposition 8.1.3. *The set of the Hermite polynomials is an orthonormal Hilbert basis in $L^2(\mathbb{R}, \gamma_1)$.*

Proof. We introduce the auxiliary analytic function

$$F : \mathbb{R}^2 \to \mathbb{R}, \quad F(t,x) := e^{-t^2/2 + tx}.$$

Since

$$F(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \Big(-\frac{t^2}{2} + tx \Big)^k,$$

for every $x \in \mathbb{R}$ the Taylor expansion of F with respect to t, centred at $t = 0$, converges for every $t \in \mathbb{R}$ and we write it as

$$\begin{aligned} F(t,x) &= e^{x^2/2} e^{-(t-x)^2/2} = e^{x^2/2} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^n}{\partial t^n} e^{-(t-x)^2/2} \bigg|_{t=0} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{x^2/2} (-1)^n \frac{d^n}{dx^n} e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(x). \end{aligned}$$

So, for t, $s \in \mathbb{R}$ we have

$$F(t,x)F(s,x) = e^{-(t^2+s^2)/2 + (t+s)x} = \sum_{n,m=0}^{\infty} \frac{t^n}{\sqrt{n!}} \frac{s^m}{\sqrt{m!}} H_n(x) H_m(x).$$

Integrating with respect to x in $\mathbb{R}$ and recalling that $\int_{\mathbb{R}} e^{\lambda x} \gamma_1(dx) = e^{\lambda^2/2}$ for every $\lambda \in \mathbb{R}$ we get

$$\int_{\mathbb{R}} F(t,x)F(s,x)\, \gamma_1(dx) = e^{-(t^2+s^2)/2} \int_{\mathbb{R}} e^{(t+s)x} \gamma_1(dx) = e^{ts} = \sum_{n=0}^{\infty} \frac{t^n s^n}{n!},$$

as well as

$$\int_{\mathbb{R}} F(t,x)F(s,x)\,\gamma_1(dx) = \sum_{n,m=0}^{\infty} \frac{t^n}{\sqrt{n!}}\frac{s^m}{\sqrt{m!}} \int_{\mathbb{R}} H_n(x)H_m(x)\,\gamma_1(dx).$$

Comparing the series gives, for every n, $m \in \mathbb{N}\cup\{0\}$,

$$\int_{\mathbb{R}} H_n(x)H_m(x)\,\gamma_1(dx) = \delta_{n,m},$$

which shows that the set of the Hermite polynomials is orthonormal.

Let now $f \in L^2(\mathbb{R},\gamma_1)$ be orthogonal to all the Hermite polynomials. Since the linear span of $\{H_k:\ k\le n\}$ is the set of all polynomials of degree $\le n$, f is orthogonal to all powers x^n. Then, all the derivatives of the holomorphic function

$$g(z) = \int_{\mathbb{R}} \exp\{ixz\}f(x)\,d\gamma_1(x)$$

vanish at $z=0$, showing that $g\equiv 0$. For $z=t\in\mathbb{R}$, $g(t)$ is nothing but (a multiple of) the Fourier transform of $x\mapsto f(x)e^{-x^2/2}$, which therefore vanishes a.e. So, $f(x)=0$ a.e., and the proof is complete. □

Next, we define d-dimensional Hermite polynomials.

Definition 8.1.4. *If $\alpha=(\alpha_1,\dots,\alpha_d)\in(\mathbb{N}\cup\{0\})^d$ is a multiindex, we define the polynomial H_α by*

$$H_\alpha(x) = H_{\alpha_1}(x_1)\cdots H_{\alpha_d}(x_d), \qquad x=(x_1,\dots,x_d)\in\mathbb{R}^d. \tag{8.1.3}$$

Proposition 8.1.5. *The system of Hermite polynomials is an orthonormal Hilbert basis in $L^2(\mathbb{R}^d,\gamma_d)$. Moreover, for every multiindex $\alpha=(\alpha_1,\dots,\alpha_d)$ the following equality holds,*

$$\Delta H_\alpha(x) - \langle x, \nabla H_\alpha(x)\rangle = -\Big(\sum_{j=1}^{d}\alpha_j\Big)H_\alpha(x). \tag{8.1.4}$$

Proof. Since γ_d is the product measure of d copies of γ_1, and every H_α is a product of one dimensional Hermite polynomials, Proposition 8.1.3 yields $\langle H_\alpha, H_\beta\rangle_{L^2(\mathbb{R}^d,\gamma_d)} = 1$ if $\alpha=\beta$ and $\langle H_\alpha,H_\beta\rangle_{L^2(\mathbb{R}^d,\gamma_d)} = 0$ if $\alpha\ne\beta$. Completeness may be shown by recurrence on d. By Proposition 8.1.3 the statement holds for $d=1$. Assume that the statement holds for $d=n-1$, and let $f\in L^2(\mathbb{R}^n,\gamma_n)$ be orthogonal to all Hermite polynomials in $\mathbb{R}^n$. The Hermite polynomials in $\mathbb{R}^n$ are all the functions of the form $H_\alpha(x_1,\dots,x_n) = H_k(x_1)H_\beta(x_2,\dots,x_n)$ with $k\in\mathbb{N}\cup\{0\}$ and $\beta\in(\mathbb{N}\cup\{0\})^{n-1}$. So, for every $k\in\mathbb{N}\cup\{0\}$ and $\beta\in(\mathbb{N}\cup\{0\})^{n-1}$ we have

$$0 = \langle f, H_\alpha\rangle_{L^2(\mathbb{R}^n,\gamma_n)} = \int_{\mathbb{R}} \left(H_k(x_1)\int_{\mathbb{R}^{n-1}} f(x_1,y)H_\beta(y)\gamma_{n-1}(dy)\right)\gamma_1(dx_1).$$

Then, the function $g(x_1)=\int_{\mathbb{R}^{n-1}} f(x_1,y)H_\beta(y)\gamma_{n-1}(dy)$ is orthogonal in $L^2(\mathbb{R},\gamma_1)$ to all H_k. By Proposition 8.1.3 it vanishes for a.e. x_1, which means that for a.e. $x_1\in\mathbb{R}$ the function $f(x_1,\cdot)$ is orthogonal, in $L^2(\mathbb{R}^{n-1},\gamma_{n-1})$, to all Hermite polynomials H_β. By the recurrence assumption, $f(x_1,y)$ vanishes for a.e. $y\in\mathbb{R}^{n-1}$.

For $d=1$ equality (8.1.4) has already been stated in Lemma 8.1.2. For $d\geq 2$ we have

$$\begin{aligned}
D_jH_\alpha(x) &= H'_{\alpha_j}(x_j)\prod_{h\neq j}H_{\alpha_h}(x_h)\\
\Delta H_\alpha(x) &= \sum_{j=1}^d H''_{\alpha_j}(x_j)\prod_{h\neq j}H_{\alpha_h}(x_h)=\sum_{j=1}^d\Big[x_jH'_{\alpha_j}(x_j)-\alpha_jH_{\alpha_j}(x_j)\Big]\prod_{h\neq j}H_{\alpha_h}(x_h)\\
&=\sum_{j=1}^d x_jD_jH_\alpha(x)-\Big(\sum_{j=1}^d\alpha_j\Big)H_\alpha(x)=\langle x,\nabla H_\alpha(x)\rangle-\Big(\sum_{j=1}^d\alpha_j\Big)H_\alpha(x).
\end{aligned}$$

□

Let us denote by $\mathcal{X}_k$ the linear span of all Hermite polynomials of degree k. It is a finite dimensional subspace of $L^2(\mathbb{R}^d,\gamma_d)$, hence it is closed. For $f\in L^2(\mathbb{R}^d,\gamma_d)$, we denote by $I_k(f)$ the orthogonal projection of f on $\mathcal{X}_k$, given by

$$I_k(f)=\sum_{|\alpha|=k}\langle f,H_\alpha\rangle H_\alpha. \tag{8.1.5}$$

We recall that if $\alpha=(\alpha_1,\dots,\alpha_n)$ then $|\alpha|=\alpha_1+\ldots+\alpha_n$, so that the degree of H_α is $|\alpha|$. By Proposition 8.1.5 we have

$$f=\sum_{k=0}^\infty I_k(f), \tag{8.1.6}$$

where the series converges in $L^2(\mathbb{R}^d,\gamma_d)$.

8.1.2 The infinite dimensional case

The Hermite polynomials in infinite dimension are the functions $x\mapsto H_\alpha(\ell_1(x),\dots,\ell_d(x))$, where $d\in\mathbb{N}$, H_α is any Hermite polynomial in $\mathbb{R}^d$, and $\ell_j\in X^*$ for $j=1,\dots,d$ and $H_\alpha=1$ if $\alpha=0$. To our aim, it is enough to consider the sequence of elements of X^* given by $\ell_j=\hat h_j$, where $\{\hat h_j:\ j\in\mathbb{N}\}$ is a fixed orthonormal basis of X^*_γ, so that $\{h_j:\ j\in\mathbb{N}\}$ is a fixed orthonormal basis of H. We pointm out that this is always possible, see Lemma 3.1.9.

For notational convenience we introduce the set Λ of multi-indices $\alpha\in(\mathbb{N}\cup\{0\})^{\mathbb{N}}$, $\alpha=(\alpha_j)$, with finite length $|\alpha|=\sum_{j=1}^\infty\alpha_j<\infty$. Λ is just the set of all $\mathbb{N}\cup\{0\}$-valued sequences, that are eventually 0.

Definition 8.1.6. *For every $\alpha\in\Lambda$, $\alpha=(\alpha_j)$, we set*

$$H_\alpha(x)=\prod_{j=1}^\infty H_{\alpha_j}(\hat h_j(x)),\quad x\in X. \tag{8.1.7}$$

Note that only a finite number of terms in the above product are different from 1. So, every H_α is a smooth function with polynomial growth at infinity, namely $|H_\alpha(x)| \leq C(1+\|x\|^{|\alpha|})$. Therefore, $H_\alpha \in L^p(X,\gamma)$ for every $p \geq 1$.

Theorem 8.1.7. *The set $\{H_\alpha :\ \alpha \in \Lambda\}$ is an orthonormal basis of $L^2(X,\gamma)$.*

Proof. Let us first show the orthogonality. Let α, β be in Λ, and let $d \in \mathbb{N}$ be such that $\alpha_j = \beta_j = 0$ for every $j > d$. We have

$$\begin{aligned}\int_X H_\alpha\, H_\beta\, d\gamma &= \int_X \prod_{j=1}^d H_{\alpha_j}(\hat{h}_j(x)) H_{\beta_j}(\hat{h}_j(x))\, \gamma(dx)\\ &= \int_{\mathbb{R}^d} \prod_{j=1}^d H_{\alpha_j}(\xi_j) H_{\beta_j}(\xi_j) \gamma_d(d\xi)\end{aligned}$$

which is equal to 1 if $\alpha_j = \beta_j$ for every j (namely, if $\alpha = \beta$), otherwise it vanishes. The statement follows.

Next, let us prove that the linear span of the H_α with $\alpha \in \Lambda$ is dense in $L^2(X,\gamma)$. By Theorem 7.4.6, the cylindrical functions of the type $f(x) = \varphi(\hat{h}_1(x), \ldots, \hat{h}_d(x))$ with $d \in \mathbb{N}$ and $\varphi \in C_b(\mathbb{R}^d, \gamma_d)$ are dense in $L^2(X,\gamma)$. So, it is sufficient to approach such functions. To this aim, we recall that the linear span of the Hermite polynomials in $\mathbb{R}^d$ is dense in $L^2(\mathbb{R}^d, \gamma_d)$, by Proposition 8.1.5; more precisely the sequence

$$\sum_{k=0}^n I_k^{(d)}(\varphi) = \sum_{k=0}^n \sum_{\alpha \in (\mathbb{N}\cup\{0\})^d,\, |\alpha|=k} \langle \varphi, H_\alpha \rangle_{L^2(\mathbb{R}^d,\gamma_d)} H_\alpha$$

converges to φ in $L^2(\mathbb{R}^d, \gamma_d)$ as $n \to \infty$. Set

$$f_n(x) := \sum_{k=0}^n \sum_{\alpha \in (\mathbb{N}\cup\{0\})^d,\, |\alpha|=k} \langle \varphi, H_\alpha \rangle_{L^2(\mathbb{R}^d,\gamma_d)} H_\alpha(\hat{h}_1(x), \ldots, \hat{h}_d(x)), \quad n \in \mathbb{N},\ x \in X.$$

Since $\gamma \circ (\hat{h}_1, \ldots, \hat{h}_d)^{-1}$ is the standard Gaussian measure γ_d in $\mathbb{R}^d$,

$$\|f - f_n\|_{L^2(X,\gamma)} = \Big\| \varphi - \sum_{k=0}^n I_k^{(d)}(\varphi) \Big\|_{L^2(\mathbb{R}^d,\gamma_d)}, \quad n \in \mathbb{N},$$

so that $f_n \to f$ in $L^2(X,\gamma)$. □

Definition 8.1.8. *For every $k \in \mathbb{N} \cup \{0\}$ we set*

$$\mathcal{X}_k = \overline{\operatorname{span}\{H_\alpha :\ \alpha \in \Lambda,\ |\alpha| = k\}},$$

where the closure is in $L^2(X,\gamma)$.

For $k = 0$, $\mathcal{X}_0$ is the subset of $L^2(X, \gamma)$ consisting of constant functions. In contrast with the case $X = \mathbb{R}^d$, for any fixed length $k \in \mathbb{N}$ there are infinitely many Hermite polynomials H_α with $|\alpha| = k$, so that $\mathcal{X}_k$ is infinite dimensional. For $k = 1$, $\mathcal{X}_1$ is the closure of the linear span of the functions $\hat{h}_j$, $j \in \mathbb{N}$, that are the Hermite polynomials H_α with $|\alpha| = 1$. Therefore, it coincides with X^*_γ.

Proposition 8.1.9. *(The Wiener Chaos decomposition)*

$$L^2(X, \gamma) = \bigoplus_{k \in \mathbb{N} \cup \{0\}} \mathcal{X}_k.$$

Proof. Since $\langle H_\alpha \, H_\beta \rangle_{L^2(X,\gamma)} = 0$ for $\alpha \neq \beta$, the subspaces $\mathcal{X}_k$ are mutually orthogonal. Moreover, they span $L^2(X, \gamma)$ by Theorem 8.1.7. □

As in finite dimension, we denote by I_k the orthogonal projection on $\mathcal{X}_k$. So,

$$I_k(f) = \sum_{\alpha \in \Lambda, \, |\alpha| = k} \langle f, H_\alpha \rangle_{L^2(X,\gamma)} H_\alpha, \quad f \in L^2(X, \gamma), \tag{8.1.8}$$

$$f = \sum_{k=0}^{\infty} I_k(f), \quad f \in L^2(X, \gamma), \tag{8.1.9}$$

where the series converge in $L^2(X, \gamma)$.

8.2 The zero-one law

We start this section by presenting an important technical notion that we need later, that of *completion* of a σ-algebra.

Definition 8.2.1. *Let $\mathscr{F}$ be a σ-algebra of subsets of X and let γ be a measure on $(X, \mathscr{F})$. The* completion *of $\mathscr{F}$ is the family*

$$\mathscr{F}_\gamma = \Big\{ E \subset X : \; \exists \, B_1, B_2 \in \mathscr{F} \;\; \text{such that } B_1 \subset E \subset B_2, \; \gamma(B_2 \setminus B_1) = 0 \Big\}.$$

We leave as an exercise to verify that $\mathscr{F}_\gamma$ is a σ-algebra. The measure γ is extended to $\mathscr{F}_\gamma$ in the natural way. From now on, unless otherwise specified, a set $E \subset X$ is called *measurable* if it belongs to the completed σ-algebra $\mathscr{B}(X)_\gamma = \mathscr{E}(X)_\gamma$. The main result of this section is the following.

Theorem 8.2.2. *If V is a measurable affine subspace of X* [(1)]*, then $\gamma(V) \in \{0, 1\}$.*

We need some preliminary results.

Proposition 8.2.3. *If $A \in \mathscr{B}(X)_\gamma$ is such that $\gamma(A + h) = \gamma(A)$ for all $h \in H$, then $\gamma(A) \in \{0, 1\}$.*

[(1)] By measurable affine subspace we mean a set $V = V_0 + x_0$, with V_0 measurable (linear) subspace and $x_0 \in X$.

Proof. Let $\{h_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of H. Then, for every $n \in \mathbb{N}$ the function

$$F(t_1,\dots,t_n) = \gamma(A - t_1h_1 + \ldots - t_nh_n) = \int_A \exp\Big\{\sum_{j=1}^n t_j\hat{h}_j(x) - \frac{1}{2}\sum_{j=1}^n t_j^2\Big\}\gamma(dx)$$

is constant. Therefore, for all $\alpha_1,\dots,\alpha_n$ not all 0 we get

$$\frac{\partial^{\alpha_1+\ldots+\alpha_n}F}{\partial t_1^{\alpha_1}\ldots\partial t_n^{\alpha_n}}(0,\dots,0) = 0.$$

Arguing as in the proof of Proposition 8.1.3 we get

$$\frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial t_1^{\alpha_1}\ldots\partial t_n^{\alpha_n}} \exp\Big\{\sum_{j=1}^n t_j\hat{h}_j(x) - \frac{1}{2}\sum_{j=1}^n t_j^2\Big\}\Big|_{t_1=\cdots=t_n=0} = H_{\alpha_1}(\hat{h}_1(x))\cdot\ldots\cdot H_{\alpha_n}(\hat{h}_n(x))$$

(where H_{α_j} are the 1-dimensional Hermite polynomials), whence

$$\int_X H_{\alpha_1}(\hat{h}_1(x))\cdot\ldots\cdot H_{\alpha_n}(\hat{h}_n(x))1\!\!1_A(x)\gamma(dx) = 0.$$

It follows that the function $1\!\!1_A$ is orthogonal to all nonconstant Hermite polynomials and then by Theorem 8.1.7 it is constant, i.e., either $1\!\!1_A = 0$ or $1\!\!1_A = 1$ a.e. □

Corollary 8.2.4. *If $A \in \mathscr{B}(X)_\gamma$ is such that $\gamma(A \setminus (A+h)) = 0$ for every $h \in H$, then $\gamma(A) \in \{0,1\}$.*

Proof. Since if $h \in H$ also $-h \in H$, we deduce that $\gamma(A\setminus(A-h)) = 0$ for all $h \in H$ and then $\gamma((A+h)\setminus A) = 0$. In conclusion

$$\gamma(A+h) = \gamma(A), \qquad \forall h \in H$$

and we conclude by applying Proposition 8.2.3. □

Corollary 8.2.5. *If f is a measurable function such that $f(x+h) = f(x)$, for all $h \in H$, then there exists $c \in \mathbb{R}$ such that $f(x) = c$ for a.e. $x \in X$.*

Proof. By Proposition 8.2.3, for every $t \in \mathbb{R}$ either $\gamma(\{x \in X : f(x) < t\}) = 1$ or $\gamma(\{x \in X : f(x) < t\}) = 0$. Since the function $t \mapsto \gamma(\{x \in X : f(x) < t\})$ is increasing, there exists exactly one $c \in \mathbb{R}$ such that $\gamma(\{x \in X : f < t\}) = 0$ for all $t < c$ and $\gamma(\{x \in X : f < t\}) = 1$ for all $t \geq c$. Then,

$$\gamma(\{x \in X : f(x) = c\}) = \lim_{n\to\infty}\gamma\Big(\Big\{x \in X : c - \frac{1}{n} \leq f(x) < c + \frac{1}{n}\Big\}\Big) = 1.$$

□

Now we prove our main theorem.

Proof. of Theorem 8.2.2 Let us assume first that V is a linear subspace. As we shall see at the end of the proof, passing to an affine subspace requires to prove the statement for non-centred measures.

Let V be a measurable linear space. If $V = X$ there is nothing to prove, so we may assume $X \setminus V \neq \emptyset$. Let us consider $X \times X$ with the product measure $\gamma \otimes \gamma$ and the projections $p_1, p_2 : X \times X \to X$,

$$p_1(x, y) = x, \qquad p_2(x, y) = y,$$

whose law is γ; let us define the sets

$$A_k = \{(x, y) \in X \times X : x \notin V\} \cap \{(x, y) \in X \times X : y + kx \in V\}, \qquad k \in \mathbb{N}.$$

If $j \neq k$, then $A_j \cap A_k = \emptyset$; indeed, if $y + jx, y + kx \in V$, $(j - k)x \in V$ and if $j \neq k$ then $x \in V$.

We notice that, by defining $\mu_k = (\gamma \otimes \gamma) \circ (kp_1 + p_2)^{-1}$,

$$\begin{aligned}
\widehat{\mu_k}(f) &= \int_X e^{if(x)} \mu_k(dx) = \int_{X\times X} e^{if(kp_1(x,y)+p_2(x,y))} (\gamma \otimes \gamma)(d(x, y)) \\
&= \int_{X\times X} e^{i\,kf(x)+if(y)} (\gamma \otimes \gamma)(d(x, y)) = \int_X e^{i\,kf(x)} \gamma(dx) \int_X e^{if(y)} \gamma(dy) \\
&= e^{-\frac{k^2}{2}\|f\|^2_{L^2(X,\gamma)}} e^{-\frac{1}{2}\|f\|^2_{L^2(X,\gamma)}} = e^{-\frac{k^2+1}{2}\|f\|^2_{L^2(X,\gamma)}}.
\end{aligned}$$

for all $f \in X^*$. On the other hand, defining $\nu_k = (\gamma \otimes \gamma) \circ (\sqrt{k^2 + 1}p_1)^{-1}$, we have

$$\begin{aligned}
\widehat{\nu_k}(f) &= \int_X e^{if(x)}\, \nu_k(dx) = \int_{X\times X} e^{if(\sqrt{k^2+1}p_1(x,y))} (\gamma \otimes \gamma)(d(x, y)) \\
&= \int_{X\times X} e^{i\sqrt{k^2+1}f(x)} (\gamma \otimes \gamma)(d(x, y)) = e^{-\frac{k^2+1}{2}\|f\|_{L^2(X,\gamma)}}.
\end{aligned}$$

So the random variables $kp_1 + p_2$ and $\sqrt{k^2 + 1}p_1$ are identically distributed, i.e., they have the same law. In addition, since $x, y \in V$ if and only if $x, y + kx \in V$ for any $k \in \mathbb{N}$, we deduce

$$\begin{aligned}
(\gamma \otimes \gamma)(A_k) =&(\gamma \otimes \gamma)(\{(x, y) \in X \times X : y + kx \in V\}) \\
&- (\gamma \otimes \gamma)(\{(x, y) \in X \times X : x \in V\} \cap \{(x, y) : y + kx \in V\}) \\
=&(\gamma \otimes \gamma)(\{(x, y) \in X \times X : \ \sqrt{1 + k^2}x \in V\}) \\
&- (\gamma \otimes \gamma)(\{(x, y) \in X \times X : x \in V\} \cap \{(x, y) \in X \times X : y \in V\}) \\
=&(\gamma \otimes \gamma)(\{(x, y) \in X \times X : x \in V\}) - (\gamma \otimes \gamma)(\{(x, y) \in X \times X : x \in V\})^2 \\
=&\gamma(V) - \gamma(V)^2.
\end{aligned}$$

Since the the sets A_k are pairwise disjoint, $(\gamma \otimes \gamma)(A_k) = 0$, and then $\gamma(V) \in \{0, 1\}$.

Let now V be an affine subspace. Then, there is x_0 such that $V_0 = V + x_0$ is a vector subspace, hence, applying the result for V_0 to the measure γ_{x_0} we obtain $\gamma(V) = \gamma_{x_0}(V + x_0) \in \{0, 1\}$. □

Remark 8.2.6. The proof of Theorem 8.2.2 is much simpler for centred measures, if we confine to linear subspaces. Indeed, let V be a measurable linear subspace. If $\gamma(V) = 0$ there is nothing to prove. If $\gamma(V) > 0$ then there is $c > 0$ such that $B_H(0, c) \subset V - V = V$, see Proposition 3.1.6. Then, $H \subset V$, $V + h = V$ for every $h \in H$ and by Proposition 8.2.3 we have $\gamma(V) = 1$.

8.3 Measurable linear functionals

In this section we give the notion of measurable linear functionals and we prove that such functions are just the elements of X^*_γ.

Definition 8.3.1 (Measurable linear functionals)**.** *We say that* $f : X \to \mathbb{R}$ *is a* measurable linear functional *or* γ–measurable linear functional *if there exist a measurable subspace* $V \subset X$ *with* $\gamma(V) = 1$ *and a* γ*-measurable function* $f_0 : X \to \mathbb{R}$ *such that* f_0 *is linear on* V *and* $f = f_0$ γ*-a.e.*

In the above definition $f = f_0$ γ-a.e., so we may modify any measurable linear functional on a negligible set in such a way that the modification is still mesurable, as the σ-algebra $\mathscr{B}(X)$ has been completed, it is defined *everywhere* on a full-measure subspace V and it is linear on V. This will be always done in what follows. As by Theorem 3.1.8(ii), which is easily checked to hold for all measurable subspaces and not only for Boler subspaces, the Cameron-Martin space H is contained in V, all measurable linear functionals will be defined everywhere and linear on H.

Example 8.3.2. Let us exhibit two simple examples of measurable linear functionals which are not continuous.

(i) Let $f : \mathbb{R}^\infty \to \mathbb{R}$ be the functional defined by

$$f(x) = \sum_{k=1}^{\infty} c_k x_k$$

where $(c_k) \in \ell^2$. Here, as usual $\mathbb{R}^\infty$ is endowed with a countable product of standard 1-dimensional Gaussian measure, see (4.1.1). Indeed, the series defining f converges γ-a.e. in $\mathbb{R}^\infty$, but only the restriction of f to $\mathbb{R}^\infty_c$ is continuous.

(ii) Let X be a Hilbert space endowed with the Gaussian measure $\gamma = \mathscr{N}(0, Q)$, where Q is a selfadjoint positive trace-class operator with eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$. Let $\{e_k : k \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of Q in X with $Qe_k = \lambda_k e_k$ for all $k \in \mathbb{N}$. Fix a sequence $(c_k) \subset \mathbb{R}$ such that the series $\sum_k c_k^2 \lambda_k$ is convergent and define the functional

$$f(x) = \sum_{k=1}^{\infty} c_k \langle x, e_k \rangle_X.$$

Then, f is a measurable linear functional on X which is not continuous if $(c_k) \notin \ell^2$, see Exercise 8.5.

We shall call *proper measurable linear functionals* the measurable linear functionals that are linear on X.

Proposition 8.3.3. *Let f be a measurable linear functional and let V be a full measure subspace such that f is linear on V. If $X \setminus V \neq \emptyset$ then there is a modification of f on the γ-negligible set $X \setminus V$ which is proper.*

Proof. If V is a complemented subspace, just put $f = 0$ on the complementary space. If not, we use the existence of a vector (or Hamel) basis in X, i.e., an infinite (indeed, uncountable) linearly independent set of generators, see [E, Theorem 1.4.5]. Notice that the existence of such a basis is equivalent to the (countable, as X is separable) axiom of choice or Zorn Lemma. Fix a Hamel basis of V, say $\mathcal{B} = \{e_\alpha : \alpha \in \mathbb{A}\}$ for a suitable set if indices $\mathbb{A}$. Then, complete $\mathcal{B}$ in order to get a basis of X and extend $f_{|V}$ setting $f = 0$ on the added generators. The extension of $f_{|V}$ is different from f on a γ-negligible set and is linear on the whole of X. □

The first result on the measurable linear functionals is the following.

Proposition 8.3.4. *If $f : X \to \mathbb{R}$ is a linear measurable functional, then its restriction to H is continuous with respect to the norm of H.*

Proof. Setting $V_n = \{f \leq n\}$, $n \in \mathbb{N}$, since $X = \cup_n V_n$, there is $n_0 \in \mathbb{N}$ such that $\gamma(V_{n_0}) > 0$. By Lemma 3.1.6 there is $r > 0$ such that $B_H(0, r) \subset V_{n_0} - V_{n_0}$, and therefore

$$\sup_{h \in B_H(0,r)} |f(h)| \leq 2n_0.$$

□

For the statement of Proposition 8.3.4 be meaningful, f has to be defined *everywhere* on the subspace V in definition 8.3.1, because H is negligible. Nevertheless, proper functionals are uniquely determined by their values on H.

Lemma 8.3.5. *Let f be a proper measurable linear functional. If $f \in X^*_\gamma$ then*

$$f(h) = [R_\gamma f, h]_H = \int_X f(x)\hat{h}(x)\,\gamma(dx), \qquad \forall h \in H. \tag{8.3.1}$$

Proof. The second equality is nothing but the definition of inner product in H. In order to prove the first one, consider a sequence $(f_n) \subset X^*$ converging to f in $L^2(X, \gamma)$ and fix $h \in H$. By (2.3.6), writing as usual $h = R_\gamma \hat{h}$, we have

$$f_n(h) = f_n(R_\gamma \hat{h}) = \int_X f_n(x)\hat{h}(x)\ \gamma(dx).$$

The right hand side converges to the right hand side of (8.3.1), hence (up to a subsequence that we do not relabel) $f_n \to f$ a.e. Then

$$L = \{x \in X : \ f(x) = \lim_{n\to\infty} f_n(x)\}$$

is a measurable linear subspace of full measure, hence L contains H thanks to Proposition 3.1.8(ii). Therefore, $f(h) = \lim_{n\to\infty} f_n(h)$ and this is true for all $h \in H$. □

Corollary 8.3.6. *If (f_n) is a sequence of proper measurable linear functionals converging to* 0 *in measure, then their restrictions $f_{n|H}$ converge to* 0 *uniformly on the bounded subsets of H.*

Proof. Let us first show that the convergence in measure defined in (1.1.4) implies the convergence in $L^2(X,\gamma)$. Indeed, if $f_n \to 0$ in measure then

$$\exp\{-\frac{1}{2}\|f_n\|^2_{L^2(X,\gamma)}\} = \hat{\gamma}(f_n) \to 1,$$

whence $\|f_n\|_{L^2(X,\gamma)} \to 0$. Therefore, by Lemma 8.3.5

$$|f_n(h)| \leq \int_X |f_n(x)||\hat{h}(x)|\,\gamma(dx) \leq \|f_n\|_{L^2(X,\gamma)}|h|_H, \qquad \forall h \in H.$$

To show that the sequence $(f_{n|H})$ converges uniformly on the bounded sets, it is enough to consider the unit ball:

$$\sup_{h\in H, |h|_H \leq 1} |f_{n|H}(h)| \leq \|f_n\|_{L^2(X,\gamma)} \to 0.$$

□

Proposition 8.3.7. *If f and g are two measurable linear functionals, then either $\gamma(\{f = g\}) = 1$ or $\gamma(\{f = g\}) = 0$. We have $\gamma(\{f = g\}) = 1$ if and only if $f = g$ on H.*

Proof. According to Proposition 8.3.3, we may assume that f and g are proper; then $L = \{f = g\}$ is a measurable vector space. By Theorem 8.2.2 either $\gamma(L) = 0$, or $\gamma(L) = 1$. If $\gamma(L) = 1$ then $H \subset L$ by Proposition 3.1.8(ii) and then $f = g$ on H. Conversely, if $f = g$ in H then the measurable function $\varphi := f - g$ verifies $\varphi(x + h) = \varphi(x)$ for every $h \in H$. By Corollary 8.2.5 $\varphi = c$ a.e., but as φ is linear, $c = 0$. □

Notice that in the proof of Proposition 8.3.7 we use the extension of the measure γ to the completed σ-algebra, because we don't know whether $L = \{f = g\}$ is a Borel vector space. Moreover, as a consequence of Proposition 8.3.7, if a measurable linear functional vanishes on a dense subspace of H then it vanishes a.e. Indeed, any measurable linear functional is continuous on H, hence if it vanishes on a dense set then it vanishes everywhere in H.

Theorem 8.3.8. *The following conditions are equivalent.*

(i) $f \in X^*_\gamma$.

(ii) There is a sequence $(f_n)_{n\in\mathbb{N}} \subset X^*$ that converges to f in measure.

(iii) f is a measurable linear functional.

Proof. (i) $\Longrightarrow$ (ii) is obvious.
(ii) $\Longrightarrow$ (iii) If $(f_n) \subset X^*$ converges to f in measure, then (up to subsequences that we do not relabel) $f_n \to f$ a.e. and therefore defining

$$V = \{x \in X : \exists \lim_{n\to+\infty} f_n(x)\},$$

V is a measurable subspace and $\gamma(V) = 1$, hence we may define also the functional

$$f_0(x) = \lim_{n\to+\infty} f_n(x), \qquad x \in V.$$

V and f_0 satisfy the conditions of Definition 8.3.1 and therefore f is a measurable linear functional.
(iii) $\Longrightarrow$ (i) Let f be a measurable linear functional and, without loss of generality, assume that f is proper. By the Fernique Theorem $f \in L^2(X,\gamma)$ and by Proposition 8.3.4, f is continuous on H. Fix an orthonormal basis $\{h_n : n \in \mathbb{N}\}$ of H and define

$$g = \sum_{n\in\mathbb{N}} f(h_n)\hat{h}_n \quad \in X^*_\gamma.$$

We already know that the elements of X^*_γ are measurable linear functionals, so g is a measurable linear functional, and by definition $f = g$ on H, so that by Proposition 8.3.7, $f = g$ γ a.e. and $f \in X^*_\gamma$. $\square$

8.4 Exercises

Exercise 8.1. Prove the equalities (8.1.2).

Exercise 8.2. Prove that the class of functions of the form $H_{\alpha_1}(\ell_1 \cdot x)\cdots H_{\alpha_d}(\ell_d \cdot x)$, where H_{α_j} are Hermite polynomials and ℓ_j are elements of $\mathbb{R}^d$, is dense in $L^2(\mathbb{R}^d, \gamma_d)$.

Exercise 8.3. Verify that the family $\mathscr{F}_\gamma$ introduced in Definition 8.2.1 is a σ-algebra. Prove also that the measure γ, extended to $\mathscr{F}_\gamma$ by $\gamma(E) = \gamma(B_1) = \gamma(B_2)$ for E, B_1, B_2 as in Definition 8.2.1, is still a measure.

Exercise 8.4. Prove that if A is a measurable set such that $A + rh_j = A$ up to γ-negligible sets with $r \in \mathbb{Q}$ and $\{h_j : j \in \mathbb{N}\}$ an orthonormal basis of H, then $\gamma(A) \in \{0, 1\}$.
Hint: Use the continuity of the map $h \mapsto \gamma(A + h)$ in H.

Exercise 8.5. Prove that the functionals f defined in Example 8.3.2 enjoy the stated properties.
Hint: For the case (ii), prove that $f \in L^2(X,\gamma)$.

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