Solutions to the exercises of Lecture 7, ISem 2015/2016

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Exercise (7.1). *Let X be a separable Banach space endowed with a centred Gaussian measure γ. Prove that for any choice $h_1, \dots, h_d \in H$, the map $P\colon X \to \mathbb{R}^d$, $P(x) \coloneqq (\hat{h}_1(x), \dots \hat{h}_d(x))$ is a Gaussian random variable with law $P_\sharp(\gamma) = \mathscr{N}(0, Q)$, where $Q_{k,j} = [h_k, h_j]_H$.*

Proof. First of all, notice that

$$\widehat{P_\sharp\gamma}(\xi) = \int_{\mathbb{R}^d} e^{ix\xi}(P_\sharp\gamma)(dx) = \int_X e^{iP(x)\cdot\xi}\gamma(dx) = \widehat{\gamma}(g),$$

where we put $g = P \cdot \xi = \sum_{k=1}^d \widehat{h}_k \xi_k \in X^*_\gamma$. Next, observe that

$$\begin{aligned}\|g\|^2_{L^2(X,\gamma)} &= \left\langle \sum_{k=1}^d \widehat{h}_k\xi_k, \sum_{j=1}^d \widehat{h}_j\xi_j \right\rangle_{L^2(X,\gamma)} \\ &= \sum_{j,k=1}^d \xi_k\xi_j \left\langle \widehat{h}_k, \widehat{h}_j \right\rangle_{L^2(X,\gamma)} = \sum_{j,k=1}^d \xi_k\xi_j \left[h_k, h_j\right]_H,\end{aligned}$$

since the map $h \mapsto \widehat{h}$ is an isometry of H into $L^2(X,\gamma)$ by Proposition 3.1.2. Now, by Proposition 2.3.5,

$$\widehat{\gamma}(g) = e^{-\frac{1}{2}\|g\|^2_{L^2(X,\gamma)}} = e^{-\frac{1}{2}\sum_{j,k=1}^d \xi_k\xi_j[h_k,h_j]_H} = e^{-\frac{1}{2}\langle Q\xi,\xi\rangle},$$

where $Q_{k,j} = [h_k, h_j]_H$. □

Exercise (7.2, Part 1). *Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a convex function. Then there exist a sequence (a_n) in $\mathbb{R}^d$ and a sequence (b_n) in $\mathbb{R}$ such that*

$$\varphi(x) = \sup_{n\in\mathbb{N}}(a_n \cdot x + b_n).$$

Proof. We will need the following two well-known results (see *Rockafellar, Convex Analysis*, Theorems 10.1 and 23.4).

1. *If $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a convex function, then φ is continuous.*

2. *If $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a convex function, then for every $y \in \mathbb{R}^d$ there exist $a_y \in \mathbb{R}^d$ and $b_y \in \mathbb{R}$ such that*

$$\varphi(x) \geq a_y \cdot x + b_y$$

for every $x \in \mathbb{R}^d$, with equality when $x = y$. In other words, the subgradient of φ is non-empty at every point.
For every $y \in \mathbb{R}^d$ take $a_y \in \mathbb{R}^d$ and $b_y \in \mathbb{R}$ such that

$$\varphi(x) \geq a_y \cdot x + b_y$$

for every $x \in \mathbb{R}^d$, with equality when $x = y$. Let then (x_n) be any enumeration of $\mathbb{Q}^d$, and call $a_n := a_{x_n}$, $b_n := b_{x_n}$. Therefore,

$$\varphi(x) \geq a_n \cdot x + b_n$$

for every $x \in \mathbb{R}^d$ and every $n \in \mathbb{N}$. Hence,

$$\varphi(x) \geq \sup_n (a_n \cdot x + b_n) =: \psi(x)$$

for every $x \in \mathbb{R}^d$, with equality on $\mathbb{Q}^d$ by construction. Now, both φ and ψ are convex, hence continuous, functions and they agree on a dense subset of $\mathbb{R}^d$; therefore they coincide on the whole of $\mathbb{R}^d$. □

Exercise (**7.2**, Part 2 - Jensen)**.** *Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a convex function, $(\Omega, X, \mathbb{P})$ a probability space, $\mathscr{G}$ a sub-σ-algebra of $\mathscr{F}$, and $X \in L^1(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^d)$; assume that $\varphi \circ X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$. Then*

$$\varphi \circ \mathbb{E}(X|\mathscr{G}) \leq \mathbb{E}(\varphi \circ X|\mathscr{G}).$$

Proof. Take a sequence (a_n) in $\mathbb{R}^d$ and a sequence (b_n) in $\mathbb{R}$ such that

$$\varphi(x) = \sup_{n \in \mathbb{N}} (a_n \cdot x + b_n)$$

for every $x \in \mathbb{R}^d$. By monotonicity and linearity, one has $\mathbb{E}(\varphi \circ X|\mathscr{G}) \geq a_n \cdot \mathbb{E}(X|\mathscr{G}) + b_n$, so that, by taking the least upper bound,

$$\varphi \circ \mathbb{E}(X|\mathscr{G}) = \sup_{n \in \mathbb{N}} (a_n \cdot \mathbb{E}(X|\mathscr{G}) + b_n) \leq \mathbb{E}(\varphi \circ X|\mathscr{G}).$$ □

Exercise (**7.3**)**.** *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathscr{G}$ a sub-σ-algebra of $\mathscr{F}$ and $X, Y \in L^1(\Omega, \mathscr{F}, \mathbb{P})$. Then the following hold:*

1. *if $\mathscr{G} = \{\emptyset, \Omega\}$, then $\mathbb{E}(X|\mathscr{G}) = \mathbb{E}[X]$;*
2. *$\mathbb{E}[\mathbb{E}(X|\mathscr{G})] = \mathbb{E}[X]$;*
3. *if $X \leq Y$, then $\mathbb{E}(X|\mathscr{G}) \leq \mathbb{E}(Y|\mathscr{G})$;*

4. *for all* $\alpha, \beta \in \mathbb{R}$, $\mathbb{E}(\alpha X + \beta Y|\mathscr{G}) = \alpha\mathbb{E}(X|\mathscr{G}) + \beta\mathbb{E}(Y|\mathscr{G})$*;*

5. *if* $\mathscr{H}$ *is a sub-*σ*-algebra of* $\mathscr{G}$*, then* $\mathbb{E}(\mathbb{E}(X|\mathscr{G})|\mathscr{H}) = \mathbb{E}(X|\mathscr{H})$*;*

6. *if* X *is* $\mathscr{G}$*-measurable, then* $\mathbb{E}(X|\mathscr{G}) = X$*;*

7. *if* $XY \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ *and* X *is* $\mathscr{G}$*-measurable, then* $\mathbb{E}(XY|\mathscr{G}) = X\mathbb{E}(Y|\mathscr{G})$*;*

8. *if* X *is independent of* $\mathscr{G}$*, then* $\mathbb{E}(X|\mathscr{G}) = \mathbb{E}[X]$*.*

Proof.

1. We just observe that $\mathbb{E}[X]$ belongs to $L^1(\Omega, \mathscr{G}, \mathbb{P})$ and trivially satisfies the equality $\mathbb{E}[\mathbb{E}[X]\chi_B] = \mathbb{E}[X\chi_B]$ for every $B \in \{\emptyset, \Omega\}$.

2. Set $Y = \mathbb{E}(X|\mathscr{G})$. Then, the definition of conditional expectation implies that $\mathbb{E}[X\chi_B] = \mathbb{E}[Y\chi_B]$ for every $\mathscr{G}$-measurable set B. Take $B = \Omega$ to get the result.

3. Choose two representatives f, g of $\mathbb{E}(X|\mathscr{G}), \mathbb{E}(Y|\mathscr{G})$ respectively, and let, for every $n \in \mathbb{N}^*$, $E_n \coloneqq \{f \geq g + \frac{1}{n}\}$; then $E_n \in \mathscr{G}$. Moreover,

$$\mathbb{E}(g\chi_{E_n}) = \mathbb{E}(Y\chi_{E_n}) \geq \mathbb{E}(X\chi_{E_n}) = \mathbb{E}(f\chi_{E_n}) \geq \mathbb{E}(g\chi_{E_n}) + \frac{1}{n}\mathbb{P}(E_n),$$

so that $\mathbb{P}(E_n) = 0$. Then

$$\{f > g\} = \bigcup_{n \in \mathbb{N}^*} E_n$$

has $\mathbb{P}$-measure zero. Therefore $f \leq g$ $\mathbb{P}$-a.s., i.e. $\mathbb{E}(X|\mathscr{G}) \leq \mathbb{E}(Y|\mathscr{G})$.

4. First notice that $\alpha\mathbb{E}(X|\mathscr{G}) + \beta\mathbb{E}(Y|\mathscr{G})$ is $\mathscr{G}$-measurable. Moreover, for every $B \in \mathscr{G}$

$$\begin{aligned}\mathbb{E}[(\alpha X + \beta Y)\chi_B] &= \alpha\mathbb{E}[X\chi_B] + \beta\mathbb{E}[Y\chi_B] \\ &= \alpha\mathbb{E}[\mathbb{E}(X|\mathscr{G})\chi_B] + \beta\mathbb{E}[\mathbb{E}(Y|\mathscr{G})\chi_B] \\ &= \mathbb{E}[(\alpha\mathbb{E}(X|\mathscr{G}) + \beta\mathbb{E}(Y|\mathscr{G}))\chi_B],\end{aligned}$$

whence the result.

5. First notice that $\mathbb{E}(\mathbb{E}(X|\mathscr{G})|\mathscr{H})$ in $\mathscr{H}$-measurable. Moreover, if $B \in \mathscr{H}$ then $B \in \mathscr{G}$, so that $\mathbb{E}[X\chi_B] = \mathbb{E}[\mathbb{E}(X|\mathscr{G})\chi_B] = \mathbb{E}[\mathbb{E}(\mathbb{E}(X|\mathscr{G})|\mathscr{H})\chi_B]$, whence the result.

6. X itself satisfies the properties of the definition of $\mathbb{E}(X|\mathscr{G})$.

7. First notice that $X\mathbb{E}(Y|\mathscr{G})$ is $\mathscr{G}$-measurable. Observe also that by Remark 7.2.2 the statement holds for every $\mathscr{G}$-measurable simple function. Let (X_n) be a sequence of $\mathscr{G}$-measurable simple functions which converges $\mathbb{P}$-almost everywhere to X such that $|X_n| \leq |X|$ $\mathbb{P}$-almost everywhere. By the dominated convergence theorem, for every $B \in \mathscr{G}$

$$\mathbb{E}[XY\chi_B] = \lim_{n\to\infty} \mathbb{E}[X_nY\chi_B] = \lim_{n\to\infty} \mathbb{E}[X_n\mathbb{E}(Y|\mathscr{G})\chi_B] = \mathbb{E}[X\mathbb{E}(Y|\mathscr{G})\chi_B],$$

whence the result.

8. Assume first that X is the characteristic function of some $E \in \mathscr{F}$. Then, for every $B \in \mathscr{G}$,

$$\begin{aligned}\mathbb{E}[X\chi_B] = \mathbb{E}[\chi_E\chi_B] &= \mathbb{P}(E \cap B) \\ &= \mathbb{P}(E)\mathbb{P}(B) = \mathbb{E}[\mathbb{E}[\chi_E]\chi_B] = \mathbb{E}[\mathbb{E}[X]\chi_B].\end{aligned}$$

Thus $\mathbb{E}(X|\mathscr{G}) = \mathbb{E}[X]$. By linearity the statement holds for any $\mathscr{F}$-measurable simple function which is a finite linear combination of characteristic functions of sets independent of $\mathscr{G}$. Now, let X be independent of $\mathscr{G}$, and let (X_n) be a sequence of $\mathscr{E}(\Omega, X)$-measurable simple functions which converges pointwise a.e. to X such that $|X_n| \leq |X|$ $\mathbb{P}$-a.e. for every $n \in \mathbb{N}$. Then, for every $B \in \mathscr{G}$, the dominated convergence theorem implies that

$$\mathbb{E}[X\chi_B] = \lim_{n\to\infty} \mathbb{E}[X_n\chi_B] = \lim_{n\to\infty} \mathbb{E}[X_n]\mathbb{E}[\chi_B] = \mathbb{E}[\mathbb{E}[X]\chi_B]. \qquad \square$$

Exercise (7.4). *Prove that, if* $\Omega = (0,1)^2$, $\mathscr{F} = \mathscr{B}((0,1)^2)$, $\mathbb{P} = \lambda_2$ *is the Lebesgue measure on* Ω, $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, *and* $\mathscr{G} = \mathscr{B}((0,1)) \times (0,1)$, *then*

$$\mathbb{E}(X|\mathscr{G})(x,y) = \int_0^1 X(x,t)d\lambda_1(t) \qquad \text{for a.e. } (x,y) \in \Omega.$$

Proof. Since $\mathscr{B}((0,1)^2) = \mathscr{B}(0,1) \otimes \mathscr{B}(0,1)$, by Fubini's theorem the function

$$Y = \int_0^1 X(\cdot,t)d\lambda_1(t) \otimes 1$$

belongs to $L^1(\Omega, \mathscr{G}, \mathbb{P})$.

Let $A \in \mathscr{G}$; then A has the form $A = A_1 \times (0,1)$ for some $A_1 \in \mathscr{B}(0,1)$, so that $\chi_A(x,y) = \chi_{A_1}(x)$ for every $y \in (0,1)$. Observe also that $\lambda_2 = \lambda_1 \otimes \lambda_1$.

Therefore,

$$\begin{aligned}
\mathbb{E}(Y\chi_A) &= \int_0^1 \int_0^1 \int_0^1 X(x,t)d\lambda_1(t)\chi_A(x,y)d\lambda_1(x)d\lambda_1(y) \\
&= \int_0^1 \int_0^1 X(x,t)d\lambda_1(t)\chi_{A_1}(x)d\lambda_1(x) \\
&= \int_0^1 \int_0^1 X(x,y)\chi_{A_1}(x)d\lambda_1(x)d\lambda_1(y) \\
&= \int_0^1 \int_0^1 X(x,y)\chi_A(x,y)d\lambda_1(x)d\lambda_1(y) = \mathbb{E}(X\chi_A).
\end{aligned}$$

The proof is then complete. □

Exercise (7.5). *Prove that, for every fixed sequence $(f_n) \subset X^*$, the family of sets*

$$\mathscr{A} = \{E = \{x \in X\colon (f_n(x)) \in B\}\colon B \in \mathscr{B}(\mathbb{R}^\infty)\}$$

is a σ-algebra.

Proof. First of all $\emptyset \in \mathscr{A}$, since $\emptyset = \{x \in X\colon (f_n(x)) \in \emptyset\}$.

Let $E = \{x \in X\colon (f_n(x)) \in B\}$ for some $B \in \mathscr{B}(\mathbb{R}^\infty)$. Then

$$E^c = \{x \in X\colon (f_n(x)) \notin B\} = \{x \in X\colon (f_n(x)) \in B^c\} \in \mathscr{A}.$$

If $\{E_j\} \subset \mathscr{A}$ and $E_j = \{x \in X\colon (f_n(x)) \in B_j\}$, then

$$\bigcup_j E_j = \{x \in X\colon \exists j\colon (f_n(x)) \in B_j\} = \{x \in X\colon (f_n(x)) \in \cup_j B_j\} \in \mathscr{A}.$$

This completes the proof. □

Exercise (7.6). *Prove that if X is an infinite dimensional Banach space and $\varphi \in C^\infty(X)$ has compact support, then $\varphi \equiv 0$.*

Proof. Assume that there is some $x \in X$ such that $\varphi(x) \neq 0$. By continuity, there exists a closed ball $\overline{B}$ such that $\varphi(x) \neq 0$ for every $x \in \overline{B}$. Thus $\overline{B}$ is contained in the support of φ. Since the latter is compact, and $\overline{B}$ is closed, $\overline{B}$ is compact. Therefore X is finite dimensional. □

Remark. In Exercise 7.6 differentiability plays no role, and the same result holds for $\varphi \in C_c(X)$.