Lecture A

Addendum to Lecture 7

Aim of this short note is to fix some problem present in Lecture 7. There we made the wrong assertion that $\mathscr{B}(X) = \mathscr{E}(X, \mathscr{H})$, where $\mathscr{H} = \{\ell_k : k \in \mathbb{N}\}$ is a subset of X^* such that $\{h_k = R_\gamma j(\ell_k) : k \in \mathbb{N}\}$ is an orthonormal basis of H.

First of all we prove a result concerning the σ–algebra $\mathscr{E}(X, \mathscr{H})$ and then we use such result to rewrite the proof of Proposition 7.4.5.

A.1 Completion of $\mathscr{E}(X, \mathscr{H})$

Theorem A.1.1. *Let X be a separable Banach space and let γ be a centred Gaussian measure on X. If $\{h_k : k \in \mathbb{N}\}$ is an orthonormal basis of H contained in $R_\gamma(X^*)$, then*

$$\mathscr{B}(X)_\gamma = \mathscr{E}(X, \mathscr{H})_\gamma$$

with $\mathscr{H} = \{\hat{h}_k : k \in \mathbb{N}\}$.

We need some preliminary result.

Lemma A.1.2. *Let γ be a centred Gaussian measure and let $x^*, y^* \in X^*$ be such that $\|j(x^*)\|_{L^2(X,\gamma)} = \|j(y^*)\|_{L^2(X,\gamma)} = 1$ and $\|j(x^*) - j(y^*)\|^2_{L^2(X,\gamma)} = \varepsilon$. Then for any $\alpha \in \mathbb{R}$*

$$\gamma(\{x^* \geq \alpha\}\Delta\{y^* \geq \alpha\}) \leq \frac{2}{\pi}\sqrt{\frac{\varepsilon}{4-\varepsilon}},$$

where $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Proof. We use essentially the fact that $x^*, y^* : X \to \mathbb{R}$ are random variables with image measures $\mathscr{N}(0, 1)$. Let us define the map $T : X \to \mathbb{R}^2$,

$$Tx = \frac{1}{\sqrt{2}}(x^*(x) + y^*(x), x^*(x) - y^*(x)).$$

Since

$$\varepsilon = \|j(x^*) - j(y^*)\|^2_{L^2(X,\gamma)} = 2(1 - \langle j(x^*), j(y^*)\rangle_{L^2(X,\gamma)}),$$

we deduce, by Exercise 2.4, that $\gamma \circ T^{-1} = \mathscr{N}(0, Q)$ with

$$Q = \begin{pmatrix} 2 - \frac{\varepsilon}{2} & 0 \\ 0 & \frac{\varepsilon}{2} \end{pmatrix}.$$

Set $E = \{x \in X : x^*(x) \geq \alpha\}$, $F = \{x \in X : y^*(x) \geq \alpha\}$. Then $E = T^{-1}(A)$, $F = T^{-1}(B)$ where

$$A = \{(u, v) \in \mathbb{R}^2 : u + v \geq \alpha\sqrt{2}\}, \quad B = \{(u, v) \in \mathbb{R}^2 : u - v \geq \alpha\sqrt{2}\},$$

and we have

$$\begin{aligned}\gamma(E\Delta F) =& \int_X |1\!\!1_E(x) - 1\!\!1_F(x)|\gamma(dx) = \int_X |1\!\!1_A(Tx) - 1\!\!1_B(Tx)|\gamma(dx)\\ =& \int_{\mathbb{R}^2} |1\!\!1_A(u, v) - 1\!\!1_B(u, v)|\mathscr{N}(0, Q)(d(u, v))\\ =& \int_{\mathbb{R}^2} |1\!\!1_{A'}(s, t) - 1\!\!1_{B'}(s, t)|\mathscr{N}(0, I_2)(d(s, t))\\ =& 2\mathscr{N}(0, I_2)(A' \setminus B'),\end{aligned}$$

where the sets A' and B' are obtained by the change of variables $s = \frac{u}{\sqrt{2-\frac{\varepsilon}{2}}}$, $t = \frac{v}{\sqrt{\frac{\varepsilon}{2}}}$ and then

$$A' = \left\{(s, t) \in \mathbb{R}^2 : t \geq -\sqrt{\frac{4-\varepsilon}{\varepsilon}}s + \frac{2\alpha}{\sqrt{\varepsilon}}\right\}, \quad B' = \left\{(s, t) \in \mathbb{R}^2 : t \leq \sqrt{\frac{4-\varepsilon}{\varepsilon}}s - \frac{2\alpha}{\sqrt{\varepsilon}}\right\}.$$

We have

$$A' \setminus B' = \left\{(s, t) \in \mathbb{R}^2 : -t\sqrt{\frac{\varepsilon}{4-\varepsilon}} < s - \frac{2\alpha}{\sqrt{4-\varepsilon}} \leq t\sqrt{\frac{\varepsilon}{4-\varepsilon}}\right\},$$

and therefore

$$\begin{aligned}\gamma(E\Delta F) =& 2\mathscr{N}(0, I_2)(A' \setminus B')\\ =& \frac{1}{\pi}\int_0^{+\infty} e^{-\frac{t^2}{2}}\left(\int_{-\sqrt{\frac{\varepsilon}{4-\varepsilon}}t+\frac{2\alpha}{\sqrt{4-\varepsilon}}}^{\sqrt{\frac{\varepsilon}{4-\varepsilon}}t+\frac{2\alpha}{\sqrt{4-\varepsilon}}} e^{-\frac{s^2}{2}}\,ds\right)dt\\ \leq& \frac{2}{\pi}\sqrt{\frac{\varepsilon}{4-\varepsilon}}\int_0^{+\infty} te^{-\frac{t^2}{2}}dt.\end{aligned}$$

□

Proposition A.1.3. *Let $x^* \in X^*$ and let (x^*_n) be a sequence in X^* such that $j(x^*_n)$ converges to $j(x^*)$ in $L^2(X, \gamma)$ and $\|j(x^*)\|_{L^2(X,\gamma)} = \|j(x^*_n)\|_{L^2(X,\gamma)} = 1$ for any $n \in \mathbb{N}$. Fixed any $\alpha \in \mathbb{R}$, set $E = \{x : x^*(x) \geq \alpha\}$, $E_n = \{x : x^*_n(x) \geq \alpha\}$. Then*

$$\lim_{n\to+\infty} \gamma(E\Delta E_n) = 0.$$

In addition, if $\mathscr{G}$ is a sub–σ algebra of $\mathscr{F}$ and $E_n \in \mathscr{G}$ for any $n \in \mathbb{N}$, then $E \in (\mathscr{G})_\gamma$.

Proof. The first assertion follows by Lemma A.1.2 since

$$\|\mathbb{1}_E - \mathbb{1}_{E_n}\|_{L^1(X,\gamma)} = \gamma(E_n \Delta E) \leq \frac{2}{\pi}\sqrt{\frac{\|j(x^*) - j(x_n^*)\|_{L^2(X,\gamma)}}{4 - \|j(x^*) - j(x_n^*)\|_{L^2(X,\gamma)}}}.$$

Let us prove the second statement. Since $\mathbb{1}_{E_n} \to \mathbb{1}_E$ in $L^1(X,\gamma)$, $\mathbb{1}_{E_n}$ is a Cauchy sequence in $L^1(X,\mathscr{G},\gamma)$. Hence, there are a subsequence and $A \in \mathscr{G}$ with $\gamma(A) = 1$ such that $\mathbb{1}_{E_n}(x)$ converges for any $x \in A$. Set $g(x) = 0$ for $x \in X \setminus A$ and

$$g(x) = \lim_{k\to+\infty} \mathbb{1}_{E_{n_k}}(x), \qquad \forall x \in A.$$

Then g is $\mathscr{G}$–measurable. Since $\mathbb{1}_{E_{n_k}}$ has values in $\{0,1\}$, g is the characteristic function of some set $F \subset A$; the measurability of g implies that $F \in \mathscr{G}$. By the Dominated Convergence Theorem, $\mathbb{1}_{E_{n_k}}$ converges to $\mathbb{1}_F$ in $L^1(X,\mathscr{G},\gamma)$ and hence also in $L^1(X,\gamma)$. On the other hand,

$$\gamma(E\Delta F) = \|\mathbb{1}_E - \mathbb{1}_F\|_{L^1(X,\gamma)} \leq \|\mathbb{1}_E - \mathbb{1}_{E_{n_k}}\|_{L^1(X,\gamma)} + \|\mathbb{1}_{E_{n_k}} - \mathbb{1}_F\|_{L^1(X,\gamma)} \qquad \forall k \in \mathbb{N}.$$

Since $\mathbb{1}_{E_n}$ converges to $\mathbb{1}_E$ in $L^1(X,\gamma)$, the right hand side vanishes as $k \to +\infty$, $\gamma(E\Delta F) = 0$ and $E \in (\mathscr{G})_\gamma$. □

We can then prove the main result.

Proof. (*of Theorem A.1.1*) Since $\mathscr{E}(X,\mathscr{H}) \subset \mathscr{B}(X)$, the inclusion

$$\mathscr{E}(X,\mathscr{H})_\gamma \subset \mathscr{B}(X)_\gamma$$

is immediate. Let us prove the reverse inclusion. Arguing as in Theorem 2.1.1, it is enough to prove that $\overline{B}(x_0,r) \in \mathscr{E}(X,\mathscr{H})_\gamma$ for any $x_0 \in X$, $r > 0$. Since there exists a sequence $x_n^* \in X^*$ such that

$$\overline{B}(x_0,r) = \bigcap_{n\in\mathbb{N}} \{x \in X : |x^*(x - x_0)| \leq r\},$$

we are reduced to show that

$$E_{x_0,r}^{x^*} := \{x \in X : x^*(x_0) - r \leq x^*(x) \leq x^*(x_0) + r\} \in \mathscr{E}(X,\mathscr{H})_\gamma$$

for any $x^* \in X^*$, $x_0 \in X$ and $r > 0$. Let us distinguish the cases $\|j(x^*)\|_{L^2(X,\gamma)} = 0$ and $\|j(x^*)\|_{L^2(X,\gamma)} \neq 0$.

If $x^* \in X^*$ is such that $\|j(x^*)\|_{L^2(X,\gamma)} = 0$, then $\gamma \circ (x^*)^{-1}$ is a Dirac measure and then

$$\gamma(E_{x_0,r}^{x^*}) \in \{0,1\}$$

depending on the fact that $0 \in [x^*(x_0) - r, x^*(x_0) + r]$ or not. In any case, $E_{x_0,r}^{x^*}$ is a null measure set modification of either the empty set or the whole space X, and then $E_{x_0,r}^{x^*} \in \mathscr{E}(X,\mathscr{H})_\gamma$.

In the case $\|j(x^*)\|_{L^2(X,\gamma)} \neq 0$, we have

$$E_{x_0,r}^{x^*} = \{x \in X : \alpha_1 \leq \bar{x}^*(x) \leq \alpha_2\} = \{\bar{x}^* \geq \alpha_1\} \cap \{-\bar{x}^* \geq -\alpha_2\},$$

where $\bar{x}^* \in X^*$ is defined by

$$\bar{x}^* = \frac{1}{\|j(x^*)\|_{L^2(X,\gamma)}} x^*$$

so that $\|j(\bar{x}^*)\|_{L^2(X,\gamma)} = 1$ and

$$\alpha_1 = \frac{x^*(x_0) - r}{\|j(x^*)\|_{L^2(X,\gamma)}}, \qquad \alpha_2 = \frac{x^*(x_0) + r}{\|j(x^*)\|_{L^2(X,\gamma)}}.$$

We then apply Proposition A.1.3 with

$$x_n^* = c_n \sum_{k=1}^{n} \langle j(\bar{x}^*), \hat{h}_k \rangle_{L^2(X,\gamma)} \ell_k,$$

where $c_n > 0$ is a normalising constant such that $\|j(x_n^*)\|_{L^2(X,\gamma)} = 1$, and then, since $j(x_n^*)$ converges to $j(x^*)$ in $L^2(X,\gamma)$, we deduce that $E_{x_0,r}^{x^*} \in \mathscr{E}(X,\mathscr{H})_\gamma$. □

A.2 Proof of Proposition 7.4.5

Proof. Let us fix $f \in L^p(X,\gamma)$. We know that for any $\varepsilon > 0$ there exists a simple function s_ε,

$$s_\varepsilon = \sum_{i=1}^{m} c_i \mathbb{1}_{A_i}, \qquad A_i \in \mathscr{B}(X), c_i \in \mathbb{R} \setminus \{0\},$$

such that $\|f - s_\varepsilon\|_{L^p(X,\gamma)} < \varepsilon$. Since $\mathscr{B}(X) \subset \mathscr{E}(X,\mathscr{H})_\gamma$, for any $i = 1, \ldots, m$ there exists $\tilde{A}_i \in \mathscr{E}(X,\mathscr{H})$ with$\gamma(A_i \Delta \tilde{A}_i) = 0$. Since $\mathscr{E}(X,\mathscr{H})$ is the σ–algebra generated by the algebra

$$\Big\{\mathscr{E}(X,F_n) : n \in \mathbb{N}, F_n = \{\ell_1, \ldots, \ell_n\}\Big\},$$

for any $i = 1, \ldots, m$ there exists n_i and $C_i \in \mathscr{E}(X, F_{n_i})$ with $\gamma(\tilde{A}_i \Delta C_i) \leq \frac{\varepsilon^p}{m^p |c_i|^p}$. The choice of the sets C_i implies that by defining

$$\tilde{s}_\varepsilon = \sum_{i=1}^{m} c_i \mathbb{1}_{C_i},$$

we have

$$\|s_\varepsilon - \tilde{s}_\varepsilon\|_{L^p(X,\gamma)} \leq \sum_{i=1}^{m} |c_i| \|\mathbb{1}_{A_i} - \mathbb{1}_{C_i}\|_{L^p(X,\gamma)}$$

$$= \sum_{i=1}^{m} |c_i| \gamma(A_i \Delta C_i)^p = \varepsilon.$$

Then, if $n = \max\{n_i : i = 1, \dots, m\}$, since $\tilde{s}_\varepsilon$ is $\mathscr{E}(X, F_n)$–measurable, by property 6. of Proposition 7.2.3 $\mathbb{E}_n \tilde{s}_\varepsilon = \tilde{s}_\varepsilon$ and then

$$\begin{aligned}\|f - \mathbb{E}_n f\|_{L^p(X,\gamma)} \leq & \|f - s_\varepsilon\|_{L^p(X,\gamma)} + \|s_\varepsilon - \tilde{s}_\varepsilon\|_{L^p(X,\gamma)} + \\ & + \|\tilde{s}_\varepsilon - \mathbb{E}_n s_\varepsilon\|_{L^p(X,\gamma)} + \|\mathbb{E}_n s_\varepsilon - \mathbb{E}_n f\|_{L^p(X,\gamma)} \\ \leq & \|f - s_\varepsilon\|_{L^p(X,\gamma)} + \|s_\varepsilon - \tilde{s}_\varepsilon\|_{L^p(X,\gamma)} + \\ & + \|\mathbb{E}_n(\tilde{s}_\varepsilon - s_\varepsilon)\|_{L^p(X,\gamma)} + \|\mathbb{E}_n(s_\varepsilon - f)\|_{L^p(X,\gamma)} \\ \leq & 2\|f - s_\varepsilon\|_{L^p(X,\gamma)} + 2\|s_\varepsilon - \tilde{s}_\varepsilon\|_{L^p(X,\gamma)} < 4\varepsilon,\end{aligned}$$

where we have used the contractivity property of the conditional expectation. The proof is then completed. □