# Lecture 6

# The classical Wiener space

In this Lecture we present the *classical Wiener space*, which is the archetype of the structure we are describing. Indeed, any triple $(X, \gamma, H)$ (where $X$ is a separable Banach space, $\gamma$ is a Gaussian measure and $H$ is the Cameron-Martin space) is called an *abstract Wiener space*. In the classical Wiener space the Banach space is that of continuous paths, $X = C([0,1])$, and all the objects involved can be described explicitly. The Gaussian measure is the Wiener measure $\gamma^W$ defined in Lecture 5, the covariance operator is the integral operator with kernel $\min\{x, y\}$ on $[0,1]^2$ and both the Cameron-Martin space $H$ and $X^*_\gamma$ are spaces of functions defined on $[0,1]$.

## 6.1 The classical Wiener space

We start by considering the measure space $(X, \mathscr{B}(X), \gamma^W)$ where $X = C([0,1])$, $\mathscr{B}(X)$ is the Borel $\sigma$–algebra on $X$ and $\gamma^W = \mathbb{P}^W \circ \widetilde{B}^{-1}$ is the measure defined in Proposition 5.2.10.

We give the following approximation result for measures in terms of Dirac measures. For every real measure $\mu \in \mathscr{M}([0,1])$ and $n \in \mathbb{N}$ we set

$$\mu_n = \mu(\{1\})\delta_1 + \sum_{i=0}^{2^n-1} \mu\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right) \delta_{\frac{i+1}{2^n}}. \tag{6.1.1}$$

**Lemma 6.1.1.** *The following statements hold:*

*(i) for every $\mu \in \mathscr{M}([0,1])$ the sequence $(\mu_n)$ converges weakly to $\mu$;*

*(ii) for every $\mu, \nu \in \mathscr{M}([0,1])$ the sequence $(\mu_n \otimes \nu)$ converges weakly to $\mu \otimes \nu$;*

*(iii) for every $\mu, \nu \in \mathscr{M}([0,1])$ the sequence $(\mu_n \otimes \nu_n)$ converges weakly to $\mu \otimes \nu$.*

*Proof.* (i) Let us fix $f \in C([0,1])$. Then

$$\int_{[0,1]} f(x)\mu_n(dx) - \int_{[0,1]} f(x)\mu(dx) = \sum_{i=0}^{2^n-1} \int_{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)} \left(f\left(\frac{i+1}{2^n}\right) - f(x)\right)\mu(dx).$$

By the uniform continuity of $f$, for every $\varepsilon > 0$ there is $n_0 > 0$ such that for $n > n_0$ we have $|f(\frac{i+1}{2^n}) - f(x)| < \varepsilon$ for every $x \in [i2^{-n}, (i+1)2^{-n}]$ and for every $i = 0, \dots, 2^n - 1$, whence for $n > n_0$

$$\Big|\sum_{i=0}^{2^n-1} \int_{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)} \Big(f\Big(\frac{i+1}{2^n}\Big) - f(x)\Big)\, \mu(dx)\Big| < \varepsilon|\mu|([0,1])$$

and $\mu_n(f) \to \mu(f)$.
(ii) Let now $f \in C([0,1]^2)$. Setting

$$\varphi_n(y) = \int_{[0,1]} f(x,y)\mu_n(dx), \qquad \varphi(y) = \int_{[0,1]} f(x,y)\mu(dx),$$

we have

$$\int_{[0,1]^2} f(x,y)(\mu_n \otimes \nu)(d(x,y)) = \int_{[0,1]} \varphi_n(y)\nu(dy).$$

For any $y \in [0,1]$, $(\varphi_n(y))$ converges to $\varphi(y)$ by statement (i). Since

$$|\varphi_n(y)| \le \|f\|_\infty |\mu|([0,1]),$$

by the Lebesgue Dominated Convergence Theorem we have

$$\lim_{n\to+\infty} \int_{[0,1]} \varphi_n(y)\nu(dy) = \int_{[0,1]} \varphi(y)\nu(dy) = \int_{[0,1]^2} f(x,y)(\mu\otimes\nu)(d(x,y)).$$

(iii) Let now $f \in C([0,1]^2)$. Let $\varphi_n$ and $\varphi$ the functions in (ii), then

$$\begin{aligned}\int_{[0,1]^2} f(x,y)(\mu_n \otimes \nu_n)(d(x,y)) &= \int_{[0,1]} \varphi_n(x)\nu_n(dx)\\ &= \int_{[0,1]} \varphi(x)\nu_n(dx) + \int_{[0,1]} (\varphi_n(x) - \varphi(x))\nu_n(dx).\end{aligned}$$

We argue as in (i): by the uniform continuity of $f$ for any $\varepsilon > 0$ there is $n_0 > 0$ such that for $n > n_0$

$$\sup_{x\in[0,1]} |\varphi_n(x) - \varphi(x)| \le \sum_{i=0}^{2^n-1} \int_{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)} \Big|f\Big(x, \frac{i+1}{2^n}\Big) - f(x,y)\Big|\, |\mu|(dy) < \varepsilon|\mu|([0,1]).$$

Using the inequality $|\nu_n|([0,1]) \le |\nu|([0,1])$, see Exercise 6.1, we have

$$\Big|\int_{[0,1]} (\varphi_n(x) - \varphi(x))\nu_n(dx)\Big| \le \varepsilon|\mu|([0,1]) \sup_{n\in\mathbb{N}} |\nu_n|([0,1]) \le \varepsilon|\nu|([0,1])\,|\mu|([0,1])$$

so that

$$\lim_{n\to+\infty} \int_{[0,1]^2} f(x,y)(\mu_n \otimes \nu_n)(d(x,y)) = \int_{[0,1]^2} f(x,y)(\mu\otimes\nu)(d(x,y)).$$

□

Statements (ii) and (iii) in Lemma 6.1.1 can be shown in a different way and for arbitrary sequences $(\mu_n)$ and $(\nu_n)$ weakly convergent to $\mu$ and $\nu$ respectively, see Exercise 6.2.

**Proposition 6.1.2.** *The characteristic function of the Wiener measure $\gamma^W$ is*

$$\widehat{\gamma^W}(\mu) = \exp\Big\{-\frac{1}{2}\int_{[0,1]^2} \min\{x,y\}(\mu\otimes\mu)(d(x,y))\Big\}, \quad \mu\in\mathscr{M}([0,1]).$$

*So, $\gamma^W$ is a Gaussian measure with mean zero and covariance operator*

$$B_{\gamma^W}(\mu,\nu) = \int_{[0,1]^2} \min\{t,s\}(\mu\otimes\nu)(d(t,s)), \qquad \mu,\nu\in\mathscr{M}([0,1]). \tag{6.1.2}$$

*Proof.* We start by considering a linear combination of two Dirac measures

$$\mu = \alpha\delta_s + \beta\delta_t$$

with $\alpha,\beta\in\mathbb{R}, s<t\in[0,1]$. Then

$$\widehat{\gamma^W}(\mu) = \int_X \exp\{i\alpha\delta_s(f)+i\beta\delta_t(f)\}\gamma^W(df) = \int_X \exp\{i\alpha f(s)+i\beta f(t)\}\gamma^W(df).$$

Since $\gamma^W = \mathbb{P}^W_A\circ\widetilde{B}^{-1}$ with $\mathbb{P}^W(A)=1$ and $\widetilde{B}$ is a version of the Brownian motion which is continous on $A$, noticing that $(\widetilde{B}_t)_{t\in[0,1]}$ and $(B_t)_{t\in[0,1]}$ have the same image measure, we obtain

$$\begin{aligned}
\widehat{\gamma^W}(\mu) &= \int_A \exp\{i\alpha\widetilde{B}(\omega)(s)+i\beta\widetilde{B}(\omega)(t))\mathbb{P}^W(d\omega) = \int_A \exp\{i\alpha\widetilde{B}_s(\omega)+i\beta\widetilde{B}_t(\omega))\mathbb{P}^W(d\omega)\\
&= \int_{\mathbb{R}^{[0,1]}} \exp\{i\alpha\widetilde{B}_s(\omega)+i\beta\widetilde{B}_t(\omega))\mathbb{P}^W(d\omega) = \int_{\mathbb{R}^{[0,1]}} \exp\{i\alpha B_s(\omega)+i\beta B_t(\omega))\mathbb{P}^W(d\omega)\\
&= \int_{\mathbb{R}^{[0,1]}} \exp\{i(\alpha+\beta)B_s(\omega)+i\beta(B_t(\omega)-B_s(\omega))\mathbb{P}^W(d\omega).
\end{aligned}$$

Since $B_s$ and $B_t - B_s$ are independent, we may write

$$\begin{aligned}
\widehat{\gamma^W}(\mu) &= \int_{\mathbb{R}^{[0,1]}} \exp\{i(\alpha+\beta)B_s(\omega)+i\beta(B_t(\omega)-B_s(\omega))\mathbb{P}^W(d\omega)\\
&= \int_{\mathbb{R}^{[0,1]}} \exp\{i(\alpha+\beta)B_s(\omega)\}\mathbb{P}^W(d\omega)\cdot\int_{\mathbb{R}^{[0,1]}} \exp\{i\beta(B_t(\omega)-B_s(\omega))\mathbb{P}^W(d\omega)\\
&= \int_{\mathbb{R}} \exp\{i(\alpha+\beta)x\}\mathscr{N}(0,s)(dx)\cdot\int_{\mathbb{R}} \exp\{i\beta y\}\mathscr{N}(0,t-s)(dy)\\
&= \exp\left\{-\frac{1}{2}(\alpha+\beta)^2 s\right\}\exp\left\{-\frac{1}{2}\beta^2(t-s)\right\} = \exp\left\{-\frac{1}{2}\Big((\alpha^2+2\alpha\beta)s+\beta^2 t\Big)\right\}.
\end{aligned}$$

We now compute the integral

$$\int_{[0,1]^2} \min\{x,y\}(\mu\otimes\mu)(d(x,y)) = \int_{[0,1]} \varphi(x)\mu(dx) = \alpha\varphi(s)+\beta\varphi(t),$$

where $\varphi(x) = \int_{[0,1]} \min\{x,y\}\mu(dy)$. We have

$$\varphi(s) = \int_{[0,1]} \min\{s,y\}\mu(dy) = \int_{[0,s]} y\mu(dy) + s\int_{(s,1]} \mu(dy) = \alpha s + \beta s$$
$$\varphi(t) = \int_{[0,1]} \min\{t,y\}\mu(dy) = \int_{[0,t]} y\mu(dy) + t\int_{(t,1]} \mu(dy) = \alpha s + \beta t,$$

whence

$$\int_{[0,1]^2} \min\{x,y\}(\mu\otimes\mu)(d(x,y)) = (\alpha^2 + 2\alpha\beta)s + \beta^2 t.$$

So, the assertion of the theorem holds if $\mu$ is a linear combination of two Dirac measures. The same assertion holds true if $\mu$ is a linear combination of a finite number of Dirac measures. In the general case we conclude by using Lemma 6.1.1. Indeed, if $\mu_n$ is the approximation of $\mu$ defined in (6.1.1), then for any $f \in X$,

$$\lim_{n\to+\infty} \mu_n(f) = \lim_{n\to+\infty} \int_{[0,1]} f(x)\mu_n(dx) = \int_{[0,1]} f(x)\mu(dx) = \mu(f).$$

Hence $\exp\{i\mu_n(f)\}$ converges to $\exp\{i\mu(f)\}$ for any $f \in X$, so by the Lebesgue Dominated Convergence Theorem

$$\begin{aligned}\widehat{\gamma^W}(\mu) &= \int_X \exp\{i\mu(f)\}\gamma^W(df) = \lim_{n\to+\infty} \int_X \exp\{i\mu_n(f)\}\gamma^W(df)\\ &= \lim_{n\to+\infty} \exp\left\{-\frac{1}{2}\int_{[0,1]^2} \min\{x,y\}(\mu_n\otimes\mu_n)(d(x,y))\right\}\\ &= \exp\left\{-\frac{1}{2}\int_{[0,1]^2} \min\{x,y\}(\mu\otimes\mu)(d(x,y))\right\}.\end{aligned}$$

Then we conclude applying Theorem 2.2.4. □

We notice that the space

$$C_0([0,1]) := \{f \in C([0,1]) : f(0) = 0\} = \delta_0^{-1}(\{0\})$$

is a closed subspace of $C([0,1])$. Since

$$\widehat{\gamma^W}(\delta_0) = \exp\left\{-\frac{1}{2}\int_{[0,1]^2} \min\{x,y\}(\delta_0\otimes\delta_0)(d(x,y))\right\} = 1,$$

$\gamma^W \circ \delta_0^{-1} = \mathscr{N}(0,0) = \delta_0$, and so

$$\gamma^W(C_0([0,1])) = (\gamma^W \circ \delta_0^{-1})(\{0\}) = 1.$$

Then $\gamma^W$ is degenerate and it is concentrated on $C_0([0,1])$.

## 6.2 The Cameron–Martin space

In order to characterise the Cameron–Martin space of $(C([0,1]), \gamma^W)$, we use the embedding $\iota : C([0,1]) \to L^2(0,1)$, $\iota(f) = f$, which is a continuous injection since

$$\|\iota(f)\|_{L^2(0,1)} \le \|f\|_\infty.$$

If we consider the image measure $\widetilde{\gamma}^W := \gamma^W \circ \iota^{-1}$ on $L^2(0,1)$, the Cameron–Martin spaces on $C([0,1])$ and on $L^2(0,1)$ are the same in the sense of Proposition 3.1.10. The fact that $L^2(0,1)$ is a Hilbert space allows us to use the results of Section 4.2.

By using the identification $(L^2(0,1))^* = L^2(0,1)$, the characteristic function of the Gaussian measure $\widetilde{\gamma}^W$ is

$$\widehat{\widetilde{\gamma}^W}(g) = \int_{L^2(0,1)} \exp\Big\{i\langle f, g\rangle_{L^2(0,1)}\Big\}\, \tilde{\gamma}^W(df) = \int_{C([0,1])} \exp\{i\langle \iota(f), g\rangle_{L^2(0,1)}\}\, \gamma^W(df).$$

Let us compute $\langle \iota(f), g\rangle_{L^2(0,1)}$. If we denote by $\iota^* : L^2(0,1) \to \mathscr{M}([0,1])$ the adjoint of $\iota$, then

$$\langle \iota(f), g\rangle_{L^2(0,1)} = \iota^*(g)(f). \tag{6.2.1}$$

Since $\iota(f)(x) = f(x)$, (6.2.1) yields

$$\int_0^1 f(x)g(x)dx = \int_{[0,1]} f(x)\iota^*(g)(dx), \qquad \forall f \in C([0,1]).$$

Hence $\iota^*(g) = g\lambda_1$, where $\lambda_1$ is the Lebesgue measure on $[0,1]$. Therefore, according to Proposition 6.1.2,

$$\widehat{\widetilde{\gamma}^W}(g) = \widehat{\gamma^W}(\iota^* g) = \exp\left\{-\frac{1}{2}\int_{[0,1]^2} \min\{x,y\} g(x)g(y)\, d(x,y)\right\}$$

so that $\widetilde{\gamma}^W$ is a Gaussian measure with covariance

$$B_{\widetilde{\gamma}^W}(f,g) = \int_{[0,1]^2} \min\{x,y\} f(x)g(y) d(x,y) = \int_0^1 Qf(y)g(y)dy,$$

where

$$Qf(y) = \int_0^1 \min\{x,y\} f(x)dx$$

is the covariance operator $Q : L^2(0,1) \to L^2(0,1)$ introduced in Section 4.2.

**Theorem 6.2.1.** *The Cameron–Martin space $H$ of $\widetilde{\gamma}^W$ on $(L^2(0,1), \mathscr{B}(L^2(0,1)))$ is*

$$H_0^1([0,1]) := \{f \in L^2(0,1) : f' \in L^2(0,1) \text{ and } f(0) = 0\}.$$

*Proof.* As the Cameron-Martin space is the range of $Q^{1/2}$, see Theorem 4.2.6, we find the eigenvalues and eigenvectors of $Q$, i.e., we look for all $\lambda \in \mathbb{R}$ and $f \in L^2(0,1)$ such that $Qf = \lambda f$. Equality $Qf = \lambda f$ is equivalent to

$$\lambda f(x) = \int_0^1 \min\{x, y\} f(y) dy = \int_0^x y f(y) dy + x \int_x^1 f(y) dy \tag{6.2.2}$$

for a.e. $x \in [0,1]$. If (6.2.2) holds, $f$ is weakly differentiable with

$$\lambda f'(x) = \int_x^1 f(y) dy.$$

Moreover, the continuous version of $f$ vanishes at 0. Hence, in its turn, $f'$ is weakly differentiable and

$$\lambda f''(x) = -f(x) \qquad \text{a.e..}$$

The continuous version of $f'$ vanishes at 1. We have proved that if $f$ is an eigenvector of $Q$ with eigenvalue $\lambda$, then $f$ is the solution of the following problem on $(0,1)$:

$$\begin{cases} \lambda f'' + f = 0, \\ f(0) = 0, \\ f'(1) = 0. \end{cases} \tag{6.2.3}$$

On the other hand, if $f$ is the solution of problem (6.2.3), integrating between $x$ and 1

$$\lambda f'(x) = \int_x^1 f(y)\, dy,$$

whence, integrating again between 0 and $x$

$$\begin{aligned} \lambda f(x) &= \int_0^x \int_t^1 f(y)\, dy\, dt = \int_0^1 \mathbb{1}_{(0,x]}(t) \int_0^1 \mathbb{1}_{(t,1]}(y) f(y)\, dy\, dt \\ &= \int_0^1 f(y) \int_0^1 \mathbb{1}_{(0,x]}(t) \mathbb{1}_{(t,1]}(y) dt\, dy = \int_0^1 f(y) \int_0^1 \mathbb{1}_{(t,1]}(x) \mathbb{1}_{(t,1]}(y) dt\, dy \\ &= \int_0^1 \min\{x, y\} f(y)\, dy = Qf(x). \end{aligned}$$

We leave as an exercise, see Exercise 6.3, the fact if $\lambda$ is an eigenvalue, then there exists $k \in \mathbb{N}$ such that $\lambda = \lambda_k$, where

$$\lambda_k = \frac{1}{\pi^2 \left(k + \frac{1}{2}\right)^2}, \qquad k \in \mathbb{N} \tag{6.2.4}$$

and $Qe_k = \lambda_k e_k$, $\|e_k\|_{L^2(0,1)} = 1$ if and only if

$$e_k(x) = \sqrt{2} \sin\Big(\frac{x}{\sqrt{\lambda_k}}\Big) = \sqrt{2} \sin\Big(\frac{2k+1}{2} \pi x\Big). \tag{6.2.5}$$

Let us now take $f \in L^2(0,1)$ and write

$$f = \sum_{k=1}^{\infty} f_k e_k, \qquad f_k = \langle f, e_k \rangle_{L^2(0,1)} \tag{6.2.6}$$

with $e_k$ be given by (6.2.5). Applying (4.2.9), we see that $f \in H$ if and only if

$$\sum_{k=1}^{\infty} f_k^2 \lambda_k^{-1} = \pi^2 \sum_{k=1}^{\infty} f_k^2 \left(\frac{2k+1}{2}\right)^2 < +\infty.$$

Such condition allows us to define the function

$$g(x) = \sqrt{2} \sum_{k=1}^{\infty} \frac{f_k}{\sqrt{\lambda_k}} \cos\left(\frac{x}{\sqrt{\lambda_k}}\right) = \pi\sqrt{2} \sum_{k=1}^{\infty} f_k \frac{2k+1}{2} \cos\left(x\pi\Big(\frac{2k+1}{2}\Big)\right)$$

and to obtain that $g \in L^2(0,1)$ is the weak derivative of $f$. Indeed, for any $\varphi \in C_c^\infty((0,1))$,

$$\begin{aligned}\int_0^1 f(x)\varphi'(x)dx &= \sum_{k=1}^{\infty} f_k \sqrt{2} \int_0^1 \sin\left(\frac{s}{\sqrt{\lambda_k}}\right) \varphi'(x)dx \\ &= -\sum_{k=1}^{\infty} \frac{f_k\sqrt{2}}{\sqrt{\lambda_k}} \int_0^1 \cos\left(\frac{x}{\sqrt{\lambda_k}}\right) \varphi(x)dx = -\int_0^1 g(x)\varphi(x)dx.\end{aligned}$$

In conclusion $f \in H^1(0,1)$ and, by (6.2.6), its continuous version vanishes at 0, whence $f \in H_0^1([0,1])$.

Eventually, from the equality

$$|f|_H = \|Q^{-1/2} f\|_X,$$

we immediatly get $|f|_H = \|f'\|_{L^2(0,1)}$. □

**Remark 6.2.2.** We have used the notation $H_0^1([0,1])$ to characterise the Cameron–Martin space; we point out that this space is not the closure of $C_c^\infty(0,1)$ in $H^1(0,1)$.

## 6.3 The reproducing kernel

In this section we characterise the reproducing kernel, both for $X = C([0,1])$ and for $X = L^2(0,1)$. To do this, we need to introduce an important tool coming from probability, the stochastic integral (the Itô integral).

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $(W_t)_{t\in[0,1]}$ be a Brownian motion, i.e., a stochastic process on $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying the conditions of Definition 5.2.4.

If $f$ is a simple function, i.e.,

$$f(t) = \sum_{i=0}^{n-1} c_i \mathbb{1}_{[t_i,t_{i+1})}(t)$$

with $c_i \in \mathbb{R}$, $0 = t_0 < \ldots < t_n = 1$, we define the random variable on $\Omega$

$$\int_0^1 f(t)dW_t(\omega) := \sum_{i=0}^{n-1} c_i\big(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\big). \tag{6.3.1}$$

We claim that

$$\mathbb{E}\left[\left(\int_0^1 f(t)dW_t\right)^2\right] = \int_0^1 f(x)^2dx. \tag{6.3.2}$$

Indeed, we have

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^1 f(t)dW_t\Big)^2\Big] &= \sum_{i,j=0}^{n-1} c_ic_j\mathbb{E}[(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})] \\
&= \sum_{i=0}^{n-1} c_i^2\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] + 2\sum_{i=1}^{n-1}\sum_{j<i} c_ic_j\mathbb{E}[(W_{t_{j+1}} - W_{t_j})(W_{t_{i+1}} - W_{t_i})] \\
&= \sum_{i=0}^{n-1} c_i^2\mathbb{E}[|W_{t_{i+1}-t_i}|^2] + 2\sum_{i=1}^{n-1}\sum_{j<i} c_ic_j\mathbb{E}[W_{t_{j+1}} - W_{t_j}]\cdot\mathbb{E}[W_{t_{i+1}} - W_{t_i}] \\
&= \sum_{i=0}^{n-1} c_i^2(t_{i+1} - t_i) = \int_0^1 |f(x)|^2dx,
\end{aligned}$$

where we have used the fact that $W_{t_{i+1}} - W_{t_i}$ is independent of $W_{t_{j+1}} - W_{t_j}$ if $j < i$ and the fact that $W_{t_{i+i}} - W_{t_i}$ has the same image measure as $W_{t_{i+1}-t_i}$, given by $\mathscr{N}(0, t_{i+1} - t_i)$.

For the next Theorem, we refer to [B, Section 2.11].

**Theorem 6.3.1** (Itô Integral)**.** *There exists a unique map $I_\Omega : L^2(0,1) \to L^2(\Omega, \mathbb{P})$ such that*

$$\mathbb{E}[|I_\Omega(f)|^2] = \int_0^1 |f(x)|^2dx, \qquad \forall f \in L^2(0,1) \tag{6.3.3}$$

*and such that*

$$I_\Omega(f) = \int_0^1 f(t)dW_t$$

*if $f$ is a simple function. Such a map is called the* Itô integral of $f$, *identity* (6.3.3) *is called* Itô isometry *and the Itô integral is denoted by*

$$I_\Omega(f) = \int_0^1 f(t)dW_t, \qquad f \in L^2(0,1).$$

*Proof.* Let $\mathscr{S}([0,1])$ be the linear subspace of $L^2(0,1)$ consisting of the simple functions. The map $I_\Omega : \mathscr{S}([0,1]) \to L^2(\Omega, \mathbb{P})$,

$$I_\Omega(f)(\omega) := \int_0^1 f(t)dW_t(\omega)$$

is a linear operator, defined on a dense subset of $L^2(0,1)$. Since it is continuous in the $L^2(0,1)$ topology by (6.3.2), it has a unique continuous extension to $L^2(0,1)$. $\square$

In order to apply Theorem 6.3.1, we need to define Brownian motions on the probability spaces $(C([0,1]), \mathscr{B}(C([0,1])), \gamma^W)$ and $(L^2(0,1), \mathscr{B}(L^2(0,1)), \widetilde{\gamma}^W)$. We leave as an exercise (see Exercise 6.4) the verification that the evaluation maps $W_t : C([0,1]) \to \mathbb{R}$ and $\widetilde{W}_t : L^2(0,1) \to \mathbb{R}$ given for continuous functions by

$$W_t(f) := f(t), \qquad \widetilde{W}_t(f) := f(t) \tag{6.3.4}$$

are indeed standard Brownian motions on $C([0,1])$ and on $L^2(0,1)$ respectively.

We pass now to the characterisation of the reproducing kernels $X^*_{\gamma^W}$ and $X^*_{\widetilde{\gamma}^W}$.

**Proposition 6.3.2.** *Let us consider the Gaussian measures* $\gamma^W$ *and* $\widetilde{\gamma}^W$ *on* $C([0,1])$ *and on* $L^2(0,1)$, *respectively. Then*

$$X^*_{\gamma^W} = I_{C([0,1])}(L^2(0,1))$$

*and*

$$X^*_{\widetilde{\gamma}^W} = I_{L^2(0,1)}(L^2(0,1)).$$

*Proof.* Let us consider the simple function

$$g(x) = \alpha 1\!\!1_{[0,s)}(x) + \beta 1\!\!1_{[0,t)} = (\alpha+\beta) 1\!\!1_{[0,s)}(x) + \beta 1\!\!1_{[s,t)}(x),$$

$s < t \in [0,1]$. Let $(W_t)$ be the Brownian motion defined in (6.3.4). Then by (6.3.1)

$$I_{C([0,1])}(g) = \int_0^1 g(t) dW_t = (\alpha+\beta)W_s + \beta(W_t - W_s) = \alpha W_s + \beta W_t.$$

Therefore,

$$I_{C([0,1])}(g)(f) = \alpha f(s) + \beta f(t)$$

for $\gamma^W$–a.e. $f \in C([0,1])$. On the other hand, setting

$$\mu = \alpha\delta_s + \beta\delta_t,$$

since $j(\mu)(f) = \alpha f(s) + \beta f(t)$ for any $f \in C([0,1])$, we obtain $I_{C([0,1])}(g) = j(\mu)$ $\gamma^W$–a.e.. Moreover, by (6.3.3)

$$\begin{aligned}\|g\|^2_{L^2(0,1)} &= \mathbb{E}\left[\Big(I_{C([0,1])}(g)\Big)^2\right] = \|I_{C([0,1])}(g)\|^2_{L^2(C([0,1]),\gamma^W)} \\ &= \|j(\mu)\|^2_{L^2(C([0,1]),\gamma^W)}.\end{aligned}$$

For any simple function $g \in \mathscr{S}([0,1])$

$$g(x) = \sum_{i=1}^n c_i 1\!\!1_{[0,t_i)}(x),$$

a similar computation yields $\|g\|_{L^2(0,1)} = \|j(\mu)\|_{L^2(C([0,1]),\gamma^W)}$ with

$$\mu = \sum_{i=1}^n c_i \delta_{t_i},$$

and $I_{C([0,1])}(g) = j(\mu)$ $\gamma^W$–a.e.. Approaching any $g \in L^2(0,1)$ by a sequence of simple functions $g_n$, $I_{C([0,1])}(g_n)$ converges to $I_{C([0,1])}(g)$ in $L^2(C([0,1]),\gamma^W)$ by the Itô isometry (6.3.3). Since $I_{C([0,1])}(g_n) \in j(\mathscr{M}([0,1]))$ for every $n \in \mathbb{N}$, we have $I_{C([0,1])}(g) \in X^*_{\gamma^W}$.

This proves that $I_{C([0,1])}(L^2(0,1)) \subset X^*_{\gamma^W}$.

For the reverse inclusion, we use Lemma 6.1.1. Let us take $\mu \in \mathscr{M}([0,1])$ and let $(\mu_n)$ be the approximating sequence defined by (6.1.1). Then $\gamma^W$–a.e. $j(\mu_n) = I_{C([0,1])}(g_n)$, where

$$\begin{aligned}
g_n(x) =& \mu(\{1\}) + \sum_{i=0}^{2^n-1} \mu\left(\Big[\frac{i}{2^n}, \frac{i+1}{2^n}\Big)\right) 1\!\!1_{[0,\frac{i+1}{2^n})}(x) \\
=& \mu(\{1\}) + \sum_{i=0}^{2^n-1} \mu\left(\Big[\frac{i}{2^n}, \frac{i+1}{2^n}\Big)\right) \sum_{j=0}^{i} 1\!\!1_{[\frac{j}{2^n},\frac{j+1}{2^n})}(x) \\
=& \sum_{j=0}^{2^n-1} \mu(\{1\}) 1\!\!1_{[\frac{j}{2^n},\frac{j+1}{2^n})}(x) + \sum_{j=0}^{2^n-1}\sum_{i=j}^{2^n-1} \mu\left(\Big[\frac{i}{2^n}, \frac{i+1}{2^n}\Big)\right) 1\!\!1_{[\frac{j}{2^n},\frac{j+1}{2^n})}(x) \\
=& \sum_{j=0}^{2^n-1} c_{n,j} 1\!\!1_{[\frac{j}{2^n},\frac{j+1}{2^n})}(x), \qquad (6.3.5)
\end{aligned}$$

where

$$c_{n,j} = \mu(\{1\}) + \sum_{i=j}^{2^n-1} \mu\left(\Big[\frac{i}{2^n}, \frac{i+1}{2^n}\Big)\right) = \mu\left(\Big[\frac{j}{2^n}, 1\Big]\right).$$

The functions $j(\mu_n)$ converge to $j(\mu)$ in $L^2(C([0,1]),\gamma^W)$ since

$$\begin{aligned}
&\|j(\mu_n) - j(\mu)\|^2_{L^2(C([0,1]),\gamma^W)} = \|j(\mu_n - \mu)\|^2_{L^2(C([0,1]),\gamma^W)} = B_{\gamma^W}(\mu_n - \mu, \mu_n - \mu) \\
&= \int_{[0,1]^2} \min\{x,y\}\big((\mu_n - \mu)\otimes(\mu_n - \mu)\big)(d(x,y)) \\
&= \int_{[0,1]^2} \min\{x,y\}(\mu_n \otimes \mu_n)(d(x,y)) - 2\int_{[0,1]^2} \min\{x,y\}(\mu_n \otimes \mu)(d(x,y)) \\
&\quad + \int_{[0,1]^2} \min\{x,y\}(\mu \otimes \mu)(d(x,y)).
\end{aligned}$$

As $\mu_n$ converges to $\mu$ weakly, $\mu_n \otimes \mu$ and $\mu_n \otimes \mu_n$ converge to $\mu \otimes \mu$ weakly, we obtain

$$\lim_{n\to+\infty} \|j(\mu_n) - j(\mu)\|^2_{L^2(C([0,1]),\gamma^W)} = 0.$$

Hence $j(\mu)$ is the limit in $L^2(C([0,1]),\gamma^W)$ of $j(\mu_n) = I_{C([0,1])}(g_n)$, whence $j(\mathscr{M}([0,1])) \subset I_{C([0,1])}(L^2(0,1))$. Hence $X^*_{\gamma^W} \subset I_{C([0,1])}(L^2(0,1))$, and this concludes the proof of the equality $X^*_{\gamma^W} = I_{C([0,1])}(L^2(0,1))$.

The proof of equality $X^*_{\widetilde{\gamma}^W} = I_{L^2(0,1)}(L^2(0,1))$ is similar. □

**Remark 6.3.3.** By Proposition 3.1.2, a function $f \in C([0,1])$ belongs to the Cameron–Martin space $H$ if and only if it belongs to the range of $R_{\gamma^W}$, namely if and only if $f = R_{\gamma^W}(I_{C([0,1])}(g))$ for some $g \in L^2(0,1)$. In this case, by (3.1.4) we have

$$|f|_H = \|I_{C([0,1])}(g)\|_{L^2(C([0,1]),\gamma^W)},$$

and by the Itô isometry (6.3.3)

$$|f|_H = \|g\|_{L^2(0,1)}.$$

The same argument holds true for $f \in L^2(0,1)$.

We close this lecture by characterising the spaces

$$R_{\gamma^W}(j(\mathscr{M}([0,1])))$$

and

$$R_{\widetilde{\gamma}^W}(j(L^2(0,1))) = Q(L^2(0,1)).$$

The latter is easier to describe. We have indeed the following result.

**Proposition 6.3.4.** *Let $\widetilde{\gamma}^W$ be the Wiener measure on $L^2(0,1)$. Then*

$$Q(L^2(0,1)) = \{u \in H^1_0([0,1]) \cap H^2(0,1) : u'(1) = 0\}$$

*and for any $f \in L^2(0,1)$, $u = Qf$ is the solution of the problem on $(0,1)$*

$$\begin{cases} u'' = -f \\ u(0) = 0, \\ u'(1) = 0. \end{cases} \tag{6.3.6}$$

*Proof.* If $u = Qf$, then

$$u(x) = Qf(x) = \int_0^1 \min\{x,y\} f(y)dy.$$

Then $u$ is weakly differentiable and

$$u'(x) = \int_x^1 f(y)dy.$$

Hence $u'$ admits a continuous version such that $u'(1) = 0$; $u'$ is also a.e. differentiable with

$$u''(x) = -f(x).$$

On the other hand, arguing as in the proof of Proposition 6.2.1, if $u$ is a solution of (6.3.6), integrating twice we obtain

$$u(x) = \int_0^1 \min\{x,y\} f(y)dy = Qf(x),$$

and this completes the proof. $\square$

To prove a similar result in the case of $\gamma^W$ on $C([0,1])$, we need the following lemma.

**Lemma 6.3.5.** *Let $v \in L^2(0,1)$ be such that $v' = \mu \in \mathscr{M}([0,1])$ in the sense of distributions, i.e.*

$$\int_0^1 v(x)\varphi'(x)dx = -\int_{[0,1]} \varphi(x)\mu(dx), \qquad \forall \varphi \in C_c^1(0,1).$$

*Then there exists $c \in \mathbb{R}$ such that for a.e. $x \in (0,1)$*

$$v(x) = \mu((0,x]) + c. \tag{6.3.7}$$

*Proof.* Let us set $w(x) = \mu((0,x])$. We claim that $w' = \mu$ is the sense of distributions. Indeed, for any $\varphi \in C_c^1(0,1)$, by the Fubini Theorem 1.1.15

$$\begin{aligned}\int_0^1 w(x)\varphi'(x)dx &= \int_0^1 \mu((0,x])\varphi'(x)dx = \int_0^1 \left(\int_{(0,1)} \mathbb{1}_{(0,x]}(y)\mu(dy)\right)\varphi'(x)dx \\ &= \int_{(0,1)} \left(\int_0^1 \mathbb{1}_{(0,x]}(y)\varphi'(x)\,dx\right)\mu(dy) = -\int_{(0,1)} \varphi(y)\mu(dy).\end{aligned}$$

As a consequence, the weak derivative of $v - w$ is zero and the conclusion holds. $\square$

**Definition 6.3.6.** *Let $v \in L^2(0,1)$ be such that $v' \in \mathscr{M}([0,1])$ in the sense of distributions. Then, writing $v(x) = \mu((0,x]) + c$ for a.e $x \in (0.1)$ as in (6.3.7), we set*

$$v(1^-) := c + \mu((0,1)).$$

We close this lecture with the following proposition.

**Proposition 6.3.7.** *Let $\gamma^W$ be the Wiener measure on $C([0,1])$. Then*

$$\begin{aligned}R_{\gamma^W}(j(\mathscr{M}([0,1]))) = \big\{u \in H_0^1([0,1]) :&\exists \mu \in \mathscr{M}([0,1]) \text{ s.t. } u'' = -\mu \text{ on } (0,1) \\ &\text{in the sense of distributions, } u'(1^-) = \mu(\{1\})\big\}\end{aligned}$$

*and $u = R_{\gamma^W}(j(\mu))$ if and only if $u$ is the solution of the following problem on $(0,1)$*

$$\begin{cases} u'' = -\mu, \\ u(0) = 0, \\ u'(1^-) = \mu(\{1\}). \end{cases} \tag{6.3.8}$$

*Proof.* Let $u = R_{\gamma^W}(j(\mu))$. Then for any $\nu \in \mathscr{M}([0,1])$ we have

$$\begin{aligned}\nu(u) = \nu(R_{\gamma^W}(j(\mu))) &= \int_{C([0,1])} j(\mu)(f)j(\nu)(f)\gamma^W(df) \\ &= \int_{C([0,1])} \int_{[0,1]} f(x)\mu(dx) \int_{[0,1]} f(y)\nu(dy)\gamma^W(df) \\ &= \int_{[0,1]^2} \left(\int_{C([0,1])} W_x(f)W_y(f)\gamma^W(df)\right)(\mu\otimes\nu)(d(x,y)) \\ &= \int_{[0,1]^2} \min\{x,y\}(\mu\otimes\nu)(d(x,y)),\end{aligned}$$

where we have used the fact that $W_t$ is a standard Brownian motion. Indeed, since if $x < y$, $W_x$ and $W_y - W_x$ are independent,

$$\begin{aligned}\int_{C([0,1])} W_x(f)W_y(f)\gamma^W(df) &= \int_{C([0,1])} W_x(f)\big(W_y(f) - W_x(f)\big)\gamma^W(df) + \\ &\quad + \int_{C([0,1])} W_x(f)^2\gamma^W(df) \\ &= \int_{C([0,1])} W_x(f)\gamma^W(df) \cdot \int_{C([0,1])} \big(W_y(f) - W_x(f)\big)\gamma^W(df) + \\ &\quad + \int_{C([0,1])} W_x(f)^2\gamma^W(df) = x.\end{aligned}$$

Then

$$u(y) = R_{\gamma^W}(j(\mu))(y) = \int_{[0,1]} \min\{x, y\}\mu(dx). \tag{6.3.9}$$

We claim that for a.e. $y \in (0,1)$

$$u'(y) = \mu((y,1]) = -\mu((0,y]) + \mu((0,1]) \tag{6.3.10}$$

and that $u' \in L^2(0,1)$. Indeed, for any $\varphi \in C^1_c(0,1)$,

$$\begin{aligned}\int_0^1 u(x)\varphi'(x)dx &= \int_{[0,1]}\int_0^1 \min\{x,y\}\varphi'(x)dx\,\mu(dy) \\ &= -\int_{[0,1]}\int_0^y \varphi(x)dx\,\mu(dy) = -\int_0^1 \varphi(x)\int_{[0,1]} \mathbb{1}_{(0,y)}(x)\mu(dy)\,dx \\ &= -\int_0^1 \varphi(x)\int_{[0,1]} \mathbb{1}_{(x,1]}(y)\mu(dy)\,dx = -\int_0^1 \varphi(x)\mu((x,1])\,dx.\end{aligned}$$

It is readily verified that $u'(x) = \mu((x,1])$ belongs to $L^2(0,1)$; in addition $u'' = -\mu$ on $(0,1)$ in the sense of distributions. Indeed for any $\varphi \in C^1_c(0,1)$

$$\begin{aligned}\int_0^1 u'(x)\varphi'(x)dx &= \int_{[0,1]}\int_0^1 \mathbb{1}_{(0,y)}(x)\varphi'(x)dx\,\mu(dy) \\ &= \int_{[0,1]} \varphi(y)\mu(dy) = \int_{(0,1)} \varphi(y)\mu(dy),\end{aligned}$$

where the last equality follows from the fact that $\varphi(0) = \varphi(1) = 0$. Comparing (6.3.10) with (6.3.7) in Lemma 6.3.5, we obtain

$$\lim_{y\to 1^-} u'(y) = \mu(\{1\}).$$

On the other hand, if $u$ is a solution of (6.3.8), then since $u'' = -\mu$, by (6.3.7) in Lemma 6.3.5 we have for a.e. $x \in (0,1)$

$$u'(x) = -\mu((0,x]) + c = \mu((x,1)) - \mu((0,1)) + c.$$

Then $u'(1^-) = \mu(\{1\})$ if and only if $c = \mu(\{1\})+\mu((0,1))$, and in this case $u'(x) = \mu((x,1])$ for a.e. $x \in (0,1)$. Integrating $u'$ between 0 and $x$, we get

$$\begin{aligned} u(x) &= \int_0^x \mu((t,1])dt = \int_0^1 \mathbb{1}_{(0,x)}(t)\mu((t,1])dt = \int_0^1 \int_{[0,1]} \mathbb{1}_{(0,x)}(t)\mathbb{1}_{(t,1]}(y)\mu(dy)\,dt \\ &= \int_{[0,1]} \int_0^1 \mathbb{1}_{(t,1]}(x)\mathbb{1}_{(t,1]}(y)dt\,\mu(dy) = \int_{[0,1]} \min\{x,y\}\mu(dy), \end{aligned}$$

and then $u = R_{\gamma^W}(j(\mu))$. This completes the proof. □

## 6.4 Exercises

**Exercise 6.1.** Let $\mu \in \mathscr{M}([0,1])$ and let $\mu_n$ be the sequence defined by (6.1.1); prove that $|\mu_n|([0,1]) \le |\mu|([0,1])$.

**Exercise 6.2.** Show that if $(\mu_n)$ is a sequence of real measures on $(X, \mathscr{F}_1)$ weakly convergent to $\mu$ and $(\nu_n)$ is a sequence of measures on $(Y, \mathscr{F}_2)$ weakly convergent to $\nu$ then the sequence $(\mu_n \otimes \nu_n)$ converges weakly to $\mu \otimes \nu$, cf. statements (ii) and (iii) in Lemma 6.1.1.

**Exercise 6.3.** Verify that the eigenvalues of $Q$ in Theorem 6.2.1 are given by (6.2.4) and that the eigenfunctions with unit norm are given by (6.2.5).

**Exercise 6.4.** Prove that the stochastic processes defined in (6.3.4) are standard Brownian motions, i.e. they satisfy Definition 5.2.4.

**Exercise 6.5.** Prove that the function $g_n$ in formula (6.3.5) is given by

$$g_n(x) = \mu\left(\left[\frac{[2^n x]}{2^n}, 1\right]\right),$$

where $[2^n x]$ is the integer part of $2^n x$. Prove that if $\mu \in \mathscr{M}([0,1])$, $j(\mu) = I_{C([0,1])}(g)$ where

$$g(x) = \mu([x,1]), \qquad \text{for a.e. } x \in [0,1].$$

**Exercise 6.6.** Prove that for any $\mu \in \mathscr{M}([0,1])$, the function

$$u(y) = \int_{[0,1]} \min\{x,y\}\mu(dx), \qquad y \in [0,1],$$

is continuous.

# Bibliography

[B] V. I. Bogachev: *Gaussian Measures.* American Mathematical Society, 1998.