ISEM Lecture 6

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Exercise 1. *Let $\mu \in \mathcal{M}([0,1])$ and let μ_n be the sequence defined by (6.1.1); prove that $|\mu_n|([0,1]) \leq |\mu|([0,1])$.*

Proof. Let $\mu \in \mathcal{M}([0,1])$ and let μ_n be the sequence defined by

$$\mu_n = \mu(\{1\})\delta_q + \sum_{i=0}^{2^n-1} \mu\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right)\delta_{\frac{i+1}{2^n}}$$

We want to prove that $|\mu_n|([0,1]) \leq |\mu|([0,1])$.

We have that

$$\begin{aligned} |\mu|([0,1]) &= \sup \sum_{h=1}^{\infty} |\mu(E_h)| \\ &\geq \sum_{i=1}^{2^n-1} \left|\mu\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right)\right| + |\mu(\{1\})| \\ &= |\mu_n|([0,1]) \end{aligned}$$

□

Exercise 2. *Show that if (μ_n) is a sequence of real measures on $(X, \mathcal{F}_1)$ weakly convergent to μ and (ν_n) is a sequence of measures on $(Y, \mathcal{F}_2)$ weakly convergent to ν then the sequence $(\mu_n \otimes \nu_n)$ converges weakly to $\mu \otimes \nu$, cf. statements (ii) and (iii) in Lemma 6.1.1.*

For general metric spaces it could be a quite difficult exercise, so one has to assume additional restrictions on X and Y. Let X and Y be complete separable metric spaces, $\mathcal{F}_1, \mathcal{F}_2$ be Borel σ-algebras. Then one can apply Prohorov theorem (see [1, Theorem 8.6.2]):

Theorem 1. *Let X be a complete separable metric space and let M be a family of Borel measures on X. Then the following conditions are equivalent:*

(i) every sequence $\{\mu_n\}_{n=1}^{\infty} \subset M$ contains a weakly convergent subsequence;

(ii) the family M uniformly bounded in the variation norm and uniformly tight, i.e. for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that $\sup_{\mu \in M} |\mu|(X \setminus K_\varepsilon) < \varepsilon$.

We also will need the following lemma, which is an easy consequence of the Stone-Weierstrass theorem:

Lemma 1. *Let X, Y be complete separable metric spaces, $K_1 \subset X, K_2 \subset Y$ be compact. Then* $\mathrm{span}\,\{f_1|_{K_1} f_2|_{K_2} : f_1 \in C_b(X), f_2 \in C_b(Y)\}$ *is dense in* $C_b(K_1 \times K_2)$.

Proof of Exercise 2. Let $f \in C_b(X \times Y)$. Fix $\varepsilon > 0$. Let $K_\varepsilon^X \subset X$, $K_\varepsilon^Y \subset Y$ be compact sets obtained by Theorem 1, namely

$$\sup_n |\mu_n|(X \setminus K_\varepsilon^X) < \varepsilon,$$

$$\sup_n |\nu_n|(Y \setminus K_\varepsilon^Y) < \varepsilon.$$

Obviously the same inequalities hold true for μ and ν (one can just extend these compact sets because μ and ν are Radon measures). Now one can find $\{f^k\}_{k=1}^\infty \subset \mathrm{span}\,\{f_1 f_2 : f_1 \in C_b(X), f_2 \in C_b(Y)\}$ such that $\|f - f^k\|_{C(K_\varepsilon^X \times K_\varepsilon^Y)} \to 0$ as $k \to \infty$. Also by redefining $f^k := f^k \wedge \|f\|_{C(X\times Y)} \vee - \|f\|_{C(X \times Y)}$ one derives $\|f^k\|_{C(X\times Y)} \leq \|f\|_{C(X\times Y)}$. We know that

$$\lim_{n\to\infty} \int_{X\times Y} f^k d(\mu_n \otimes \nu_n) = \int_{X \times Y} f^k d(\mu \otimes \nu)$$

for each fixed $k \geq 1$. Let $k \geq 1$ be such that $\|f - f^k\|_{C(K_\varepsilon^X \times K_\varepsilon^Y)} \leq \varepsilon$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \int_{X\times Y} f d(\mu_n \otimes \nu_n) &\leq 2\varepsilon \|f\|_{C(X\times Y)} + \limsup_{n\to\infty} \int_{K_\varepsilon^X \times K_\varepsilon^Y} f d(\mu_n \otimes \nu_n) \\
&\leq \varepsilon(2\|f\|_{C(X\times Y)} + K) + \limsup_{n\to\infty} \int_{K_\varepsilon^X \times K_\varepsilon^Y} f^k d(\mu_n \otimes \nu_n) \\
&\leq \varepsilon(3\|f\|_{C(X\times Y)} + K) + \limsup_{n\to\infty} \int_{X\times Y} f^k d(\mu_n \otimes \nu_n) \\
&\leq \varepsilon(3\|f\|_{C(X\times Y)} + K) + \int_{X\times Y} f^k d(\mu \otimes \nu) \\
&\leq \varepsilon(3\|f\|_{C(X\times Y)} + 2K) + \int_{X\times Y} f d(\mu \otimes \nu),
\end{aligned}$$

where $K = \sup_n |\mu_n|(X)|\nu_n|(Y) \vee |\mu|(X)|\nu|(Y)$. (The supremum exists by Theorem 1.) Getting $\varepsilon \to 0$ yields

$$\limsup_{n\to\infty} \int_{X\times Y} f d(\mu_n \otimes \nu_n) \leq \int_{X\times Y} f d(\mu \otimes \nu).$$

The same type of inequality can be proven for $\liminf_{n\to\infty} \int_{X\times Y} f d(\mu_n \otimes \nu_n)$, so

$$\lim_{n\to\infty} \int_{X\times Y} f d(\mu_n \otimes \nu_n) = \int_{X\times Y} f d(\mu \otimes \nu),$$

which means that $\mu \otimes \nu$ is a weak limit of $\mu_n \otimes \nu_n$.

□

Remark 1. *The exercise holds true for so-called Prohorov spaces, i.e. for metric spaces, in which Prohorov theorem holds true. We refer the reader to [1, Chapter 8.10(ii)].*

Exercise 3. *Verify that the eigenvalues of Q in Theorem 6.2.1 are given by (6.2.4) and that the eigenfunctions with unit norm are given by (6.2.5).*

Proof. We have to solve the following equation:

$$\begin{cases} \lambda f'' + f = 0, \\ f(0) = 0, \\ f'(1) = 0. \end{cases}$$

Suppose that $f(x) = e^{cx}$ is a solution. Then:

$$\lambda c^2 e^{cx} + e^{cx} = 0.$$

If $\lambda < 0$, then the solution has the form $f(x) = c_1 e^{\frac{1}{\sqrt{-\lambda}}x} + c_2 e^{-\frac{1}{\sqrt{-\lambda}}x}$, $x \in (0,1)$, and following the initial conditions one easily derives that $c_1 = c_2 = 0$. So let $\lambda \geq 0$. Then one has that $f(x) = c_1 \cos(\frac{x}{\sqrt{\lambda}}) + c_2 \sin(\frac{x}{\sqrt{\lambda}})$. If we now substitute the beginning condition $f(0) = 0$, then we get: $0 = f(0) = c_1$. Now we look at $f'(1) = 0$. Suppose that c_2=1. Then

$$f'(x) = -\frac{1}{\sqrt{\lambda}} \cdot \cos(\frac{x}{\sqrt{\lambda}}),$$
$$f'(1) = -\frac{1}{\sqrt{\lambda}} \cdot \cos(\frac{1}{\sqrt{\lambda}}).$$

So

$$f'(1) = -\frac{1}{\sqrt{\lambda}} \cdot \cos(\frac{1}{\sqrt{\lambda}}) = 0 \tag{1}$$

This holds true only for $\lambda_k = \frac{1}{\pi^2 \cdot (k+\frac{1}{2})^2}$, $k \in \mathbb{N}$. □

Exercise 4. *Prove that the stochastic processes defined in (6.3.4) are standard Brownian motions, i.e. Definition 5.2.4.*

Before solving the exercise, we recall Definition 5.2.4.

Definition 1 (Definition 5.2.4). *A real valued standard Brownian motion on* $[0,1]$ *is a stochastic process* $(B_t)_{t\in[0,1]}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

(1.) $B_0 = 0$ *almost surely;*

(2.) for any $t, s \in [0,1], s < t$, *the law of both random variables* $B_t - B_s$ *and* B_{t-s} *is equal to* $\mathcal{N}(0, t-s)$*;*

(3.) for any $0 \leq s < t$, B_s *and* $B_t - B_s$ *are independent.*

Proof of Exercise 4. We start with the proof that $(C[0,1], \mathcal{B}(C[0,1]), \gamma^W)$, and maps $W_t : C[0,1] \to \mathbb{R}$, defined by $W_t(f) = f(t)$ defines a standard Brownian motion.

As the defining properties of standard Brownian motion are phrased in terms of there finite-dimensional distributions, we will calculate the corresponding characteristic functions. Exercise 1.6 and 1.7, will then show that indeed, (1.)-(3.) are satisfied.

We recall the result from Proposition 6.1.2. For $\mu \in \mathcal{M}([0,1])$, we have

$$\hat{\gamma}_W(\mu) = \exp\left\{-\frac{1}{2}\int_{[0,1]} x \wedge y \; \mu \otimes \mu(dx, dy)\right\}. \tag{2}$$

(2.): Fix $t, s \in [0,1], s < t$, we prove that $W_t(f) - W_s(f)$ and $W_{t-s}(f)$ have a $\mathcal{N}(0, t-s)$ distribution. It suffices to prove the first claim. The second claim follows from the first one as we can relabel $t = t - s$ and $s = 0$, noting the first claim also implies that $W_0(f) = 0$ almost surely. This in turn also proves (1.).

Thus, we prove that $W_t(f) - W_s(f)$ has a $\mathcal{N}(0, t-s)$ distribution. Fix some $\xi \in \mathbb{R}$. We calculate the characteristic function $\widehat{W_t(f)}(\xi)$:

$$\hat{W}_t(\xi) = \int e^{i\langle \xi, W_t(f) - W_s(f)\rangle} \gamma_W(df)$$

Note that $W_t : C([0,1]) \to \mathbb{R}$ is a continuous linear map. Therefore, we can consider its adjoint $W_t^* : \mathbb{R} \to \mathcal{M}([0,1])$. By the definition of the adjoint: $\langle \xi, W_t(f)\rangle = \langle W_t^*(\xi), f\rangle := \int f \, dW_t^*(\xi)$, we see that $W_t^*(\xi) = \xi\delta_t$. Using this representation, we find

$$\begin{aligned}
\hat{W}_t(\xi) &= \int e^{i\langle \xi, W_t(f) - W_s(f)\rangle} \gamma_W(df) \\
&= \int e^{i\langle W_t^*(\xi) - W_s^*(\xi), f\rangle} \gamma_W(df) \\
&= \hat{\gamma}_W(W_t^*(\xi) - W_s^*(\xi))
\end{aligned}$$

Evaluating (2) for $\mu = W_t^*(\xi) - W_s^*(\xi)$, we find

$$\begin{aligned}
\hat{\gamma}_W(\xi(\delta_t - \delta_s)) &= \exp\left\{-\frac{\xi^2}{2}\int_{[0,1]} x \wedge y \; (\delta_t - \delta_s) \otimes (\delta_t - \delta_s)(dx, dy)\right\} \\
&= \exp\left\{-\frac{\xi^2}{2}(t - 2s + s)\right\} \\
&= \exp\left\{-\frac{\xi^2}{2}(t - s)\right\}.
\end{aligned}$$

Recalling that this equals to the characteristic function of a $\mathcal{N}(0, t-s)$ random variable, we see that (2.) is satisfied by Exercise 1.6 and 1.7.

We proceed with the verification of (3.). For this we determine the two distribution of the vector $(W_s(f), W_t(f) - W_s(f))$ in $\mathbb{R}^2$. We will show that the characteristic function of this vector agrees with the characteristic function of the product distribution of a normal $\mathcal{N}(0, s)$ variable, with the distribution of a $\mathcal{N}(0, t_s)$ random variable. The result follows then from Exercise 1.6 and 1.7.

Fix $\xi_1, \xi_2 \in \mathbb{R}$. We calculate the characteristic function of the 2-d vector:

$$\begin{aligned}
&\int e^{i\langle \xi_1, W_s(f)\rangle + i\langle \xi_2, W_t(f) - W_s(f)\rangle} \gamma_W(df) \\
&= \int e^{i\langle \xi_1\delta_s + \xi_2(\delta_t - \delta_s), f\rangle} \gamma_W(df) \\
&= \hat{\gamma}_W(\xi_1\delta_s + \xi_2(\delta_t - \delta_s)) \\
&= \hat{\gamma}_W((\xi_1 - \xi_2)\delta_s + \xi_2\delta_t)
\end{aligned}$$

By (2), we obtain that

$$
\begin{aligned}
&\hat{\gamma}_W((\xi_1-\xi_2)\delta_s+\xi_2\delta_t)\\
&=\exp\left\{-\frac{1}{2}\int_{[0,1]} x\wedge y\ ((\xi_1-\xi_2)\delta_s+\xi_2\delta_t)\otimes((\xi_1-\xi_2)\delta_s+\xi_2\delta_t)(dx,dy)\right\}\\
&=\exp\left\{-\frac{1}{2}\left((\xi_1-\xi_2)^2 s+2(\xi_1-\xi_2)\xi_2 s+\xi_2^2 t\right)\right\}\\
&=\exp\left\{-\frac{1}{2}\left(\xi_1^2 s+\xi_2^2(t-s)\right)\right\}\\
&=\exp\left\{-\frac{1}{2}s\xi_1^2\right\}\exp\left\{-\frac{1}{2}(t-s)\xi_2^2\right\}
\end{aligned}
$$

So indeed, the characteristic function of the vector $(W_s(f), W_t(f)-W_s(f))$ in $\mathbb{R}^2$ equals the characteristic function of the product distribution $(\mathcal{N}(0,s),\mathcal{N}(0,t-s))$. As a consequence, $W_s(f)$ and $W_t(f)-W_s(f)$ are independent.

We proceed with the verification that the collection of maps $\tilde{W}_t(f)$ defines standard Brownian motion on $L^2[0,1]$. As was mentioned in the discussion board, the definition of $\tilde{W}_t(f)=f(t)$ has some issues. First of all, as we work on $L_2[0,1]$ the evaluation at a point is not well defined. However, Jürgen Voigt and Mattia Calzi have pointed out on the discussion board that $C[0,1]$ is a Borel subset of $L_2[0,1]$. As such, we can define

$$
\tilde{W}_t(f)=\begin{cases} f(t) & \text{if } f\in C[0,1]\\ 0 & \text{otherwise.}\end{cases}
$$

This is indeed well-defined as these maps are measurable from $L_2[0,1]$ into $\mathbb{R}$. This follows because for every fixed t, we have that for $f\in C[0,1]$ that

$$
\frac{1}{[(t+\varepsilon)\wedge 1]-[(t-\varepsilon)\vee 0]}\int_{t-\varepsilon\vee 0}^{(t+\varepsilon)\wedge 1} f(s)\,ds\to f(t).
$$

The maps on the left hand side are continuous from $L_2[0,1]$ to $\mathbb{R}$ and converge point-wise as $\varepsilon\downarrow 0$ on $C[0,1]$ to the evaluation map $W_t(f)$.
Using these definitions, we find that the law of $\tilde{W}_t(f)$ on $\mathbb{R}$ is given by the push-forward of $\tilde{\gamma}_W$: $(\tilde{W}_t)_\#(\tilde{\gamma}_W)$. But $\tilde{\gamma}_W$ is the push forward of γ_W under ι: $(\tilde{W}_t)_\#\iota_\#(\gamma_W)$. By definition, however, $\tilde{W}_t\circ\iota=W_t$. Thus we find that the law of $\tilde{W}_t(f)$ on $\mathbb{R}$ equals the law of $W_t(f)$ on $\mathbb{R}$. Similarly, we find that all finite dimensional distributions agree to those obtained via $\{W_s\}_{0\le s\le 1}$. Thus (1.) – (3.) follow from the first part of this exercise. □

Exercise 5. *(i) Prove that the function g_n in formula (6.3.5) is given by*

$$
g_n(x)=\mu\left(\left[\frac{[2^n x]}{2^n},1\right]\right),
$$

where $[2^n x]$ is the integer part of $2^n x$.

(ii) Prove that if $\mu\in\mathcal{M}([0,1])$, then $j(\mu)=I_{C[0,1]}(g)$, where $g(x):=\mu([x,1])$.

Proof of Exercise 5. (i) Let $x \in [0,1)$, say $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ with $k \in \{0,1,\dots,2^n-1\}$. Then $2^n x \in [k,k+1)$, so that $[2^n x] = k$, and

$$g_n(x) \overset{(6.3.5)}{=} \sum_{j=0}^{2^n-1} \mu\left(\left[\frac{j}{2^n}, 1\right]\right) 1_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)}(x) = \mu\left(\left[\frac{k}{2^n}, 1\right]\right) = \mu\left(\left[\frac{[2^n x]}{2^n}, 1\right]\right).$$

(ii) Put

$$\mu_n := \sum_{j=0}^{2^n-1} c_{n,j}(\delta_{\frac{j+1}{2^n}} - \delta_{\frac{j}{2^n}}), \qquad n \in \mathbb{N}.$$

Then (as in the beginning of the proof of Proposition 6.3.5) $I(g_n) = j(\mu_n)$ for all $n \in \mathbb{N}$. Furthermore, (from the end of the proof of Proposition 6.3.5 we know that) $j(\mu) = \lim_{n\to\infty} j(\mu_n)$ in $L^2(C[0,1],\gamma^W)$. Since $I_{C[0,1]} : L^2(0,1) \longrightarrow L^2(C[0,1],\gamma^W)$ is continuous (being an isometry by Theorem 6.3.1), it thus is enough to show that $g = \lim_{n\to\infty} g_n$ in $L^2(0,1)$.
In order to show that $g = \lim_{n\to\infty} g_n$ in $L^2(0,1)$, we check the conditions of the Lebesgue dominated convergence theorem. As a consequence of the formula from (i) we have the domination $|g_n| \le |\mu|([0,1])$ for all $n \in \mathbb{N}$. To see that $g = \lim_{n\to\infty} g_n$ pointwise almost everywhere, let $x \in [0,1)$. Then $\frac{[2^n x]}{2^n} \nearrow x$ as $n \to \infty$; indeed, if $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ with $k \in \{0,1,\dots,2^n-1\}$, then $2^n x \in [k,k+1)$ and $2^{n+1}x \in [2k, 2k+2)$ so that $\frac{[2^n x]}{2^n} = \frac{k}{2^n} = \frac{2k}{2^{n+1}} \le \frac{[2^{n+1}x]}{2^{n+1}}$ and $\left|\frac{[2^n x]}{2^n} - x\right| < 2^{-n}$. So $\left\{\left[\frac{[2^n x]}{2^n}, 1\right]\right\}_{n\in\mathbb{N}}$ is a decreasing sequence of intervals whose intersection is $[x,1]$, from which it follows that

$$g(x) = \mu([x,1]) = \lim_{n\to\infty} \mu\left(\left[\frac{[2^n x]}{2^n}, 1\right]\right) \overset{(i)}{=} \lim_{n\to\infty} g_n(x).$$

This completes the proof. □

Exercise 6. *Prove that for any $\mu \in \mathcal{M}([0,1])$, the function*

$$u(y) = \int_{[0,1]} \min\{x,y\}\mu(dx), \quad y \in [0,1]$$

is continuous.

Proof of Exercise 6. For any $y, z \in [0,1]$ (w.l.o.g. assume $y \le z$), we consider

$$\begin{aligned}
|u(y) - u(z)| &= \left| \int_{[0,1]} \min\{x,y\}\mu(dx) - \int_{[0,1]} \min\{x,z\}\mu(dx) \right| \\
&= \left| \int_{[0,1]} (\min\{x,y\} - \min\{x,z\})\,\mu(dx) \right| \\
&= \left| \int_{[0,y]} (x-x)\,\mu(dx) + \int_{(y,z]} (y-x)\,\mu(dx) + \int_{(z,1]} (y-z)\,\mu(dx) \right| \\
&= \left| \int_{(y,z]} (y-x)\,\mu(dx) + \int_{(z,1]} (y-z)\,\mu(dx) \right|.
\end{aligned}$$

Now, by Exercise 1.2 and the non-positivity of the two integrands on the corresponding domains, we can easily give the following upper bounds

$$\begin{aligned} |u(y) - u(z)| &\leq \left| \int_{(y,z]} (y-x) \left(\mu^+ + \mu^-\right)(dx) + \int_{(z,1]} (y-z) \left(\mu^+ + \mu^-\right)(dx) \right| \\ &\leq \sup_{x \in (y,z]} |y-x| \cdot |\mu|((y,z]) + |y-z| \cdot |\mu|((z,1]) \\ &\leq 2\,|y-z| \cdot |\mu|([0,1]), \end{aligned}$$

where we simply used the definition of total variation $|\mu|$ to obtain the last inequality. Since $\mu \in \mathcal{M}([0,1])$ is a finite measure, $\|\mu\|_{TV} := |\mu|([0,1]) < \infty$; hence for any fixed $\varepsilon > 0$, up to choose $\delta := \varepsilon / (2\|\mu\|_{TV})$, we can conclude that for any $y, z \in [0,1]$ such that $|y-z| < \delta$, then $|u(y) - u(z)| < \varepsilon$, i.e. u is (uniformly) continuous. Moreover, u is Lipschitz-continuous with Lipschitz constant $L := 2\|\mu\|_{TV}$. □

References

[1] V. I. Bogachev. *Measure theory. Vol. II.* Springer-Verlag, Berlin, 2007.