Lecture 5

The Brownian motion

In this and in the next Lecture we present a very important example: the classical Wiener space, which is to some extent the basic and main reference example of the theory. To do this, we introduce the Wiener measure and we define the Brownian motion. With these tools, in the next Lecture we shall define the stochastic integral and we shall use it to characterise the reproducing kernel X^*_γ when γ is the Wiener measure on $X = C([0,1])$, the Banach space of real valued continuos functions. For the material of this chapter we refer the reader for instance to the books [Ba, D].

5.1 Some notions from Probability Theory

In this section we recall a few notions of probability theory. As in Definition 1.1.2, a probability is nothing but a positive measure $\mathbb{P}$ on a measurable (or *probability*) space $(\Omega, \mathscr{F})$ such that $\mathbb{P}(\Omega) = 1$.

Any measurable $\mathbb{R}^d$-valued function defined on a a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called *random variable*. Usually, random variables are denoted by the last letters of the alphabet.

Using the image measure, we call $\mathbb{P} \circ X^{-1}$ the *law* of the $\mathbb{R}^d$-valued random variable $X : \Omega \to \mathbb{R}^d$. The law of a random variable is obviously a probability measure.

Given a real valued random variable $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, we denote by

$$\mathbb{E}[X] = \int_\Omega X d\mathbb{P}$$

the average or the *expectation* of X. We also define the variance of the real random variable X, in case $X \in L^2(\Omega, \mathscr{F}, \mathbb{P})$, as

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_\Omega (X - \mathbb{E}[X])^2 d\mathbb{P}.$$

Let us introduce the notion of *stochastic process*.

Definition 5.1.1. *A stochastic process* $(X_t)_{t\in I}$ *on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *indexed on the interval* $[0,1]$ *is a function* $X : [0,1]\times\Omega \to \mathbb{R}$ *such that for any* $t \in [0,1]$ *the function* $X_t(\cdot) = X(t,\cdot)$ *is a random variable on* $(\Omega, \mathscr{F}, \mathbb{P})$.

We give now the notion of independence, both for sets and for functions. Notice that a measurable set is often called *event* in the present context.

Definition 5.1.2 (Independence)**.** *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space. Two sets or events* $A, B \in \mathscr{F}$ *are independent if*

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B).$$

Two sub-σ*-algebras* $\mathscr{F}_1, \mathscr{F}_2$ *of* $\mathscr{F}$ *are independent if any set* $A \in \mathscr{F}_1$ *is independent of any set* $B \in \mathscr{F}_2$*, that is*

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B), \qquad \forall A \in \mathscr{F}_1, \forall B \in \mathscr{F}_2.$$

Given a real random variable X *and* $\mathscr{F}'$ *sub-*σ*-algebra contained in* $\mathscr{F}$*, we say that* X *is independent of* $\mathscr{F}'$ *if the* σ*-algebras* $\sigma(X)$ *and* $\mathscr{F}'$ *are independent* [(1)]. *Two random variables* X *and* Y *are independent if* $\sigma(X)$ *and* $\sigma(Y)$ *are independent. Two stochastic processes* $(X_t)_{t\in I}$ *and* $(Y_t)_{t\in I}$ *are independent if* $\sigma(X_t)$ *and* $\sigma(Y_t)$ *are independent for any* $t \in I$.

One of the first properties of independence is expressed in the following

Proposition 5.1.3. *Let* X *and* Y *be two independent real random variables on* $(\Omega, \mathscr{F}, \mathbb{P})$. *If* $X, Y, X\cdot Y \in L^1(\Omega, \mathscr{F}, \mathbb{P})$*, then*

$$\mathbb{E}[X\cdot Y] = \mathbb{E}[X]\cdot\mathbb{E}[Y].$$

Proof. Splitting both X and Y in positive and negative part, it is not restrictive to assume that X and Y are nonnegative. Let us consider two sequences of simple functions $(s_i)_{i\in\mathbb{N}}, (s'_i)_{i\in\mathbb{N}} \subset \mathcal{S}_+$ such that s_i is $\sigma(X)$-measurable and s'_i is $\sigma(Y)$-measurable for any $i \in \mathbb{N}$, and such that

$$\mathbb{E}[X] = \lim_{i\to+\infty}\mathbb{E}[s_i], \qquad \mathbb{E}[Y] = \lim_{i\to+\infty}\mathbb{E}[s'_i].$$

We have

$$s_i = \sum_{h=1}^{n_i} c_{i,h}\mathbb{1}_{A_{i,h}}, \quad s'_i = \sum_{h=1}^{m_i} c'_{i,k}\mathbb{1}_{A'_{i,k}}$$

with $A_{i,h} \in \sigma(X)$, $A'_{i.k} \in \sigma(Y)$. Then $(s_i{\cdot}s'_i)_{i\in\mathbb{N}}$ is a sequence of simple functions converging

(1) We recall that $\sigma(X)$ is the σ–algebra generated by the sets $\{\omega \in \Omega : X(\omega) < a\}$ with $a \in \mathbb{R}$.

to $X \cdot Y$ and then, by independence

$$
\begin{aligned}
\mathbb{E}[X \cdot Y] &= \lim_{i\to+\infty} \mathbb{E}[s_i \cdot s_i'] = \lim_{i\to+\infty} \sum_{h=1}^{n_i}\sum_{k=1}^{m_i} c_{i,h}c'_{i,k}\mathbb{E}[\mathbb{1}_{A_{i,h}} \cdot \mathbb{1}_{A'_{i,k}}] \\
&= \lim_{i\to+\infty} \sum_{h=1}^{n_i}\sum_{k=1}^{m_i} c_{i,h}c'_{i,k}\mathbb{P}(A_{i,h} \cap A'_{i,k}) \\
&= \lim_{i\to+\infty} \sum_{h=1}^{n_i}\sum_{k=1}^{m_i} c_{i,h}c'_{i,k}\mathbb{P}(A_{i,h}) \cdot \mathbb{P}(A'_{i,k}) \\
&= \lim_{i\to+\infty} \sum_{h=1}^{n_i} c_{i,h}\mathbb{P}(A_{i,h}) \sum_{k=1}^{m_i} c'_{i,k}\mathbb{P}(A'_{i,k}) \\
&= \lim_{i\to+\infty} \mathbb{E}[s_i] \cdot \mathbb{E}[s_i'] = \mathbb{E}[X] \cdot \mathbb{E}[Y].
\end{aligned}
$$

□

In the same way it is possible to prove the next result; we leave its verification as an exercise.

Corollary 5.1.4. *Let X and Y be two independent real random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ and let $f, g : \mathbb{R} \to \mathbb{R}$ be two Borel functions. If $f(X), g(Y), f(X) \cdot g(Y) \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, then*

$$\mathbb{E}[f(X) \cdot g(Y)] = \mathbb{E}[f(X)] \cdot \mathbb{E}[g(Y)].$$

Remark 5.1.5. Using Corollary 5.1.4, it is possible to prove that two random variables X and Y are independent if and only if $\mathbb{P} \circ (X, Y)^{-1} = (\mathbb{P} \circ X^{-1}) \otimes (\mathbb{P} \circ Y^{-1})$, see Exercise 5.1. As a consequence, Lemma 3.1.7 can be rephrased saying that for every $f, g \in X^*$, the elements $j(f), j(g)$ are orthogonal in X^*_γ iff they are independent.

5.2 The Wiener measure $\mathbb{P}^W$ and the Brownian motion

We start by considering the space $\mathbb{R}^{[0,1]}$, the set of all real valued functions defined on $[0, 1]$. We introduce the σ–algebra $\mathscr{F}$ generated by the sets

$$\{\omega \in \mathbb{R}^{[0,1]} : P_F(\omega) \in B\},$$

where $F = \{t_1, \dots, t_m\}$ is any finite set contained in $[0, 1]$, $B \in \mathscr{B}(\mathbb{R}^m)$ and $P_F : \mathbb{R}^{[0,1]} \to \mathbb{R}^m$ is defined by

$$P_F(\omega) = (\omega(t_1), \dots, \omega(t_m)).$$

We denote by $\mathscr{C}_F$ the σ–algebra $P_F^{-1}(\mathscr{B}(\mathbb{R}^m))$, and we define a measure μ_F on $\mathscr{C}_F$ by setting, in the case $0 < t_1 < \ldots < t_m$

$$\mu_F(A) = \frac{1}{(2\pi)^{\frac{m}{2}}\sqrt{t_1(t_2 - t_1) \cdot \ldots \cdot (t_m - t_{m-1})}} \int_{P_F(A)} e^{-\frac{x_1^2}{2t_1} + \ldots - \frac{(x_m - x_{m-1})^2}{2(t_m - t_{m-1})}} dx;$$

in the case $0 = t_1 < \ldots < t_m$

$$\mu_F(A) = \frac{1}{(2\pi)^{\frac{m-1}{2}} \sqrt{t_2 \cdot \ldots \cdot (t_m - t_{m-1})}} \int_{(P_F(A))_0} e^{-\frac{x_2^2}{2t_2} + \ldots - \frac{(x_m - x_{m-1})^2}{2(t_m - t_{m-1})}} dx'$$

where

$$(P_F(A))_0 = \{x' \in \mathbb{R}^{m-1} : (0, x') \in P_F(A)\}.$$

For $F = \{0\}$, we set $\mu_{\{0\}} = \delta_0$, the Dirac measure at 0. In this way we have defined a family of measures μ_F on the σ–algebras $\mathscr{C}_F$.

We shall use the following result to extend the family of measure μ_F to a unique probability measure on $(\mathbb{R}^{[0,1]}, \mathscr{F})$. It is known as the Daniell–Kolmogorov extension theorem, that we present only in the version we need for our purposes. Its proof relies on the following basic results.

Proposition 5.2.1. *Let μ be a real valued finitely additive set function on an algebra $\mathscr{A}$. Then μ is countably additive on $\mathscr{A}$ if and only if it is continuous at $\emptyset$, i.e.*

$$\lim_{n \to +\infty} \mu(A_n) = 0$$

for every decreasing sequence of sets $(A_n) \subset \mathscr{A}$ such that $\cap_{n \in \mathbb{N}} A_n = \emptyset$.

Theorem 5.2.2. *Let μ be a real valued countably additive set function on an algebra $\mathscr{A}$. Then μ extends to a unique finite measure on the σ-algebra generated by $\mathscr{A}$.*

The proof of Proposition 5.2.1 is left as an exercise, see Exercise 5.2. For Theorem 5.2.2 we refer to [D, Theorems 3.1.4, 3.1.10]).

Theorem 5.2.3 (Daniell–Kolmogorov extension)**.** *There exists a unique probability measure $\mathbb{P}^W$, called the* Wiener measure *on $(\mathbb{R}^{[0,1]}, \mathscr{F})$ such that for every finite $F \subset [0,1]$, $\mathbb{P}^W(A) = \mu_F(A)$ if $A \in \mathscr{C}_F$.*

Proof. We notice that if $F' = F \cup \{t_{m+1}\}$ with $t_m < t_{m+1} \leq 1$, then for any $B \in \mathscr{B}(\mathbb{R}^m)$, $P_F^{-1}(B) = P_{F'}^{-1}(B \times \mathbb{R})$, so that

$$\mu_F(P_F^{-1}(B)) = \mu_{F'}(P_{F'}^{-1}(B \times \mathbb{R})).$$

This argument can be generalised to the case $F \subset G \subset [0,1]$, F and G finite sets with cardinality m and n respectively, to conclude that if $A = P_F^{-1}(B) = P_G^{-1}(B')$, $B \in \mathscr{B}(\mathbb{R}^m)$, $B' \in \mathscr{B}(\mathbb{R}^n)$, then $\mu_F(A) = \mu_G(A)$. So, for $A \in \mathscr{C}_F$, we can set

$$\mathbb{P}^W(A) := \mu_F(A).$$

The set function $\mathbb{P}^W$ is defined on the algebra

$$\mathscr{A} = \bigcup_{F \subset [0,1] \text{ finite}} \mathscr{C}_F;$$

it is finitely additive since if $A \in \mathscr{C}_F$ and $B \in \mathscr{C}_G$ are two disjoint sets, then $A \cup B \in \mathscr{C}_{F\cup G}$ and

$$\begin{aligned}\mathbb{P}^W(A\cup B) &= \mu_{F\cup G}(A\cup B) = \mu_{F\cup G}(A) + \mu_{F\cup G}(B) = \mu_F(A) + \mu_G(B)\\ &= \mathbb{P}^W(A) + \mathbb{P}^W(B).\end{aligned}$$

Moreover, $\mathbb{P}^W(\mathbb{R}^{[0,1]}) = 1$. To extend $\mathbb{P}^W$ to the σ–algebra $\mathscr{F}$, we apply Proposition 5.2.1 and Theorem 5.2.2. Let us prove that $\mathbb{P}^W$ is continuous at $\emptyset$. Assume by contradiction that there are $\varepsilon > 0$ and a sequence $(A_n) \in \mathscr{A}$ of decreasing sets whose intersection is empty, such that

$$\mathbb{P}^W(A_n) > \varepsilon, \qquad \forall n \in \mathbb{N}.$$

Without loss of generality, we may assume that $A_n = P_{F_n}^{-1}(B_n)$ with F_n containing n points and $B_n \in \mathscr{B}(\mathbb{R}^n)$, and that $F_n \subset F_{n+1}$. Denote by $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection such that $\pi_n \circ P_{F_{n+1}} = P_{F_n}$. Since each measure $\mu_{F_n} \circ P_{F_n}^{-1}$ is a Radon measure in $\mathbb{R}^n$, for every $n \in \mathbb{N}$ there is a compact set $K_n \subset B_n$ such that $\mathbb{P}^W(A_n \setminus C_n) < \frac{\varepsilon}{2^n}$, where $C_n = P_{F_n}^{-1}(K_n)$. By replacing K_{n+1} by $\tilde{K}_{n+1} = K_{n+1} \cap \pi_n^{-1}(K_n)$, we get $\tilde{K}_{n+1} \subset \pi_n^{-1}(\tilde{K}_n)$. In order to see that the $\tilde{K}_n$ are nonempty, we bound from below their measure. Setting as before $\tilde{C}_n = P_{F_n}^{-1}(\tilde{K}_n)$, we have

$$\begin{aligned}\mu_{F_n} \circ P_{F_n}^{-1}(\tilde{K}_n) &= \mathbb{P}^W(\tilde{C}_n) = \mathbb{P}^W(A_n) - \mathbb{P}^W(A_n \setminus \tilde{C}_n)\\ &= \mathbb{P}^W(A_n) - \mathbb{P}^W\Big(\bigcup_{k=1}^n A_n \setminus C_k\Big) \ge \mathbb{P}^W(A_n) - \mathbb{P}^W\Big(\bigcup_{k=1}^n A_k \setminus C_k\Big)\\ &\ge \varepsilon - \sum_{k=1}^n \frac{\varepsilon}{2^k} > 0.\end{aligned}$$

Therefore, for any $n \in \mathbb{N}$ we can pick an element

$$x^{(n)} = (x_1^{(n)}, \dots, x_n^{(n)}) \in \tilde{K}_n.$$

Since $\tilde{K}_n \subset \pi_{n-1}^{-1}(\tilde{K}_{n-1})$, the sequence $(x_1^{(n)})$ is contained in $\tilde{K}_1$, then up to subsequences, it converges to $y_1 \in \tilde{K}_1$, i.e. there exists $(x_1^{(k_n)})$ converging to y_1. The sequence $(x_1^{(k_n)}, x_2^{(k_n)})$ is contained in $\tilde{K}_2$, then up to subsequences, there exists y_2 such that it converges to $(y_1, y_2) \in \tilde{K}_2$. Iterating the procedure, we can define a sequence (y_n) such that

$$(y_1, \dots, y_n) \in \tilde{K}_n, \qquad \forall n \in \mathbb{N}.$$

Then

$$P_{F_n}^{-1}(\{(y_1, \dots, y_n)\}) \subset \tilde{C}_n \subset A_n, \qquad \forall n \in \mathbb{N},$$

hence

$$S := \{\omega \in \mathbb{R}^{[0,1]} :\ \omega(t_j) = y_j\ \forall j \in \mathbb{N}\} \subset \bigcap_{n=1}^\infty A_n$$

which is a contradiction, as $S \neq \emptyset$. Therefore, $\mathbb{P}^W$ is continuous at $\emptyset$. By Proposition 5.2.1, $\mathbb{P}^W$ is countably additive, and by Theorem 5.2.2 it has a unique extension (still denoted by $\mathbb{P}^W$) to the σ-algebra $\mathscr{F}$ generated by $\mathscr{A}$. $\square$

Once the Wiener measure has been defined, we give a formal definition of the Brownian motion.

Definition 5.2.4 (Standard Brownian motion)**.** *A real valued standard Brownian motion on* $[0,1]$ *is a stochastic process* $(B_t)_{t\in[0,1]}$ *on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *such that:*

1. $B_0 = 0$ *almost surely;*
2. *for any* $t, s \in [0,1]$*,* $s < t$*, the law of both random variables* $B_t - B_s$ *and* B_{t-s} *is equal to* $\mathscr{N}(0, t-s)$*;*
3. *for any* $0 \le s < t$*,* B_s *and* $B_t - B_s$ *are independent.*

An explicit construction of a Brownian motion is in the following proposition.

Proposition 5.2.5 (Construction and properties of Brownian motion)**.** *Given the probability space* $(\mathbb{R}^{[0,1]}, \mathscr{F})$*, the family of functions* $B_t : \mathbb{R}^{[0,1]} \to \mathbb{R}$ *defined by*

$$B_t(\omega) = \omega(t), \qquad t \in [0,1]$$

is a real valued standard Brownian motion on $[0,1]$*.*

Proof. The proof relies on the equalities

$$B_t(\omega) = \omega(t) = P_{\{t\}}(\omega).$$

First of all we notice that for any $t \in [0,1]$

$$B_t^{-1}(\mathscr{B}(\mathbb{R})) = \mathscr{C}_{\{t\}}.$$

Then B_t is $\mathscr{F}$-measurable and $(B_t)_{t\in[0,1]}$ is a stochastic process.
By definition of the Wiener measure, we have

$$\mathbb{P}^W(B_0 \in A) = \mu_{\{0\}}(P_{\{0\}}^{-1}(A)) = \delta_0(A), \qquad \forall A \in \mathscr{B}(\mathbb{R}),$$

and then $B_0 = 0$, $\mathbb{P}^W$–almost surely. Let us now compute $\mathbb{P}^W \circ B_{t-s}^{-1}$, for $t > s$. For every Borel set $A \subset \mathbb{R}$,

$$\begin{aligned}\mathbb{P}^W(B_{t-s} \in A) =& \mathbb{P}^W(P_{\{t-s\}}^{-1}(A)) = \mu_{\{t-s\}}(P_{\{t-s\}}^{-1}(A)) \\ =& \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{x^2}{2(t-s)}} dx = \mathscr{N}(0, t-s)(A).\end{aligned}$$

On the other hand, if we define $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) := y - x$, then

$$\{\omega \in \mathbb{R}^{[0,1]} : B_t(\omega) - B_s(\omega) \in A\} = \{\omega \in \mathbb{R}^{[0,1]} : \omega(t) - \omega(s) \in A\}$$

$$= \{\omega \in \mathbb{R}^{[0,1]} : h(P_{\{s,t\}}(\omega)) \in A\} = P_{\{s,t\}}^{-1}(h^{-1}(A)).$$

Hence

$$\begin{aligned}\mathbb{P}^W(\{B_t - B_s \in A\}) =& \mu_{\{s,t\}}(P^{-1}_{\{s,t\}}(h^{-1}(A))) = \frac{1}{2\pi\sqrt{s(t-s)}}\int_{h^{-1}(A)} e^{-\frac{x^2}{2s}+\frac{(y-x)^2}{2(t-s)}}\,dxdy\\ =&\frac{1}{\sqrt{2\pi s}}\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi(t-s)}}\int_{(h^{-1}(A))_x} e^{-\frac{(y-x)^2}{2(t-s)}}\,dy\right)e^{-\frac{x^2}{2s}}\;dx\end{aligned}$$

where

$$(h^{-1}(A))_x = \{y \in \mathbb{R} : (x,y) \in h^{-1}(A)\} = \{y \in \mathbb{R} : y - x \in A\} = A + x.$$

As a consequence,

$$\begin{aligned}\frac{1}{\sqrt{2\pi(t-s)}}\int_{(h^{-1}(A))_x} e^{-\frac{(y-x)^2}{2(t-s)}}\,dy &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{A+x} e^{-\frac{(y-x)^2}{2(t-s)}}\,dy\\ &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{A} e^{-\frac{z^2}{2(t-s)}}\,dz\\ &= \mathscr{N}(0, t-s)(A),\end{aligned}$$

and therefore $\mathbb{P}^W(\{B_t - B_s \in A\}) = \mathscr{N}(0, t-s)(A)$.

In order to verify independence, we fix $0 < s < t$ and $A_1, A_2 \in \mathscr{B}(\mathbb{R})$. Then

$$\{B_s \in A_1\} = P^{-1}_{\{s\}}(A_1) = P^{-1}_{\{s,t\}}(A_1 \times \mathbb{R}),$$

and

$$\{B_t - B_s \in A_2\} = P^{-1}_{\{s,t\}}(h^{-1}(A_2)),$$

so we have

$$\begin{aligned}\mathbb{P}^W(\{B_s \in A_1\} \cap \{B_t - B_s \in A_2\}) &= \mathbb{P}^W(P^{-1}_{\{s,t\}}((A_1 \times \mathbb{R}) \cap h^{-1}(A_2))\\ &= \mu_{\{s,t\}}(P^{-1}_{\{s,t\}}((A_1 \times \mathbb{R}) \cap h^{-1}(A_2))\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{s(t-s)}}\int_{(A_1\times\mathbb{R})\cap h^{-1}(A_2)} e^{-\frac{x^2}{2s}-\frac{(y-x)^2}{2(t-s)}}\,dxdy\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{s(t-s)}}\int_{\mathbb{R}} e^{-\frac{x^2}{2s}}\left(\int_{((A_1\times\mathbb{R})\cap h^{-1}(A_2))_x} e^{-\frac{(y-x)^2}{2(t-s)}}\,dy\right)dx\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{s(t-s)}}\int_{A_1} e^{-\frac{x^2}{2s}}\left(\int_{A_2+x} e^{-\frac{(y-x)^2}{2(t-s)}}\,dy\right)dx\\ &= \frac{1}{\sqrt{2\pi s}}\int_{A_1} e^{-\frac{x^2}{2s}}\,dx\frac{1}{\sqrt{2\pi(t-s)}}\int_{A_2} e^{-\frac{z^2}{2(t-s)}}\,dz\\ &= \mathbb{P}^W(\{B_s \in A_1\}) \cdot \mathbb{P}^W(\{B_t - B_s \in A_2\}).\end{aligned}$$

□

Now, we have a measure on $(\mathbb{R}^{[0,1]}, \mathscr{F})$, but we are looking for a measure on a separable Banach space. We now show how to define the measure $\mathbb{P}^W$ on $C([0,1])$; this is not immediate because $C([0,1])$ does not belong to $\mathscr{F}$. To avoid this problem, the main point is to prove that the Brownian motion (B_t) can be modified in a convenient way to obtain a process with continuous trajectories. We prove something more, namely that the trajectories are Hölder continuous for $\mathbb{P}^W$ a.e. $\omega \in \mathbb{R}^{[0,1]}$.

We need the following useful lemma. We recall that the limsup of a sequence of sets (A_n) is defined by

$$\limsup_{n\to+\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

and it is the set of points ω such that $\omega \in A_n$ for infinitely many $n \in \mathbb{N}$.

Lemma 5.2.6 (Borel-Cantelli)**.** *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $(A_n)_{n\in\mathbb{N}} \subset \mathscr{F}$ be a sequence of measurable sets. If*

$$\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) < +\infty,$$

then

$$\mathbb{P}\left(\limsup_{n\to+\infty} A_n\right) = 0.$$

Proof. We define the sets

$$B_n := \bigcup_{k\geq n} A_k.$$

Then $B_{n+1} \subset B_n$ for every n, and setting

$$B := \bigcap_{n\in\mathbb{N}} B_n = \limsup_{n\to+\infty} A_n,$$

by the continuity property of measures along monotone sequences (see Remark 1.1.3)

$$\mathbb{P}(B) = \lim_{n\to+\infty} \mathbb{P}(B_n).$$

On the other hand,

$$\begin{aligned}\lim_{n\to+\infty} \mathbb{P}(B_n) &= \lim_{n\to+\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right)\\ &\leq \lim_{n\to+\infty} \sum_{k\geq n} \mathbb{P}(A_k) = 0.\end{aligned}$$

□

Now we state and prove the Kolmogorov continuity theorem; we need the notion of version of a stochastic process. Given two stochastic processes $X_t, \widetilde{X}_t$, $t \in [0,1]$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we say that $\widetilde{X}_t$ is a *version* of X_t if

$$\mathbb{P}(\{X_t \neq \widetilde{X}_t\}) = 0, \qquad \forall t \in [0,1].$$

We use also the Chebychev's inequality, whose proof is left as Exercise 5.3. For any $\beta > 0$, for any measurable function f such that $|f|^\beta \in L^1(\Omega, \mu)$ we have

$$\mu(\{|f| \geq \lambda\}) \leq \frac{1}{\lambda^\beta} \int_\Omega |f|^\beta d\mu \qquad \forall \lambda > 0.$$

Theorem 5.2.7 (Kolmogorov continuity Theorem)**.** *Let $(X_t)_{t\in[0,1]}$ be a stochastic process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and assume that there exist $\alpha, \beta > 0$, such that*

$$\mathbb{E}[|X_t - X_s|^\beta] \leq C|t-s|^{1+\alpha}, \quad t,\ s \in [0,1].$$

Then there exists a version $(\widetilde{X}_t)_{t\in[0,1]}$ such that the map $t \mapsto \widetilde{X}_t(\omega)$ is γ–Hölder continuous for any $\gamma < \frac{\alpha}{\beta}$ and for $\mathbb{P}$–a.e. $\omega \in \Omega$.

Proof. Let us define

$$\mathscr{D}_n = \left\{ \frac{k}{2^n} : k = 0, \dots, 2^n \right\}, \qquad \mathscr{D} = \bigcup_{n\in\mathbb{N}} \mathscr{D}_n.$$

We compute the measures of the sets

$$A_n = \left\{ \max_{1\leq k \leq 2^n} \left| X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}} \right| \geq \frac{1}{2^{\gamma n}} \right\};$$

using Chebychev's inequality. We have

$$\begin{aligned}
\mathbb{P}(A_n) &= \mathbb{P}\left(\bigcup_{k=1}^{2^n} \left\{ \left| X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}} \right| \geq \frac{1}{2^{\gamma n}} \right\} \right) \\
&\leq \sum_{k=1}^{2^n} \mathbb{P}\left(\left\{ \left| X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}} \right| \geq \frac{1}{2^{\gamma n}} \right\} \right) \\
&\leq \sum_{k=1}^{2^n} 2^{\gamma n \beta} \mathbb{E}\left[\left| X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}} \right|^\beta \right] \\
&\leq C \sum_{k=1}^{2^n} 2^{\gamma n \beta} \left| \frac{k}{2^n} - \frac{k-1}{2^n} \right|^{1+\alpha} \\
&= C 2^{-n(\alpha - \gamma\beta)}.
\end{aligned}$$

As a consequence we obtain that the series

$$\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) \leq C \sum_{n\in\mathbb{N}} 2^{-n(\alpha-\gamma\beta)}$$

is convergent if $\gamma < \frac{\alpha}{\beta}$. In this case, by the Borel-Cantelli Lemma 5.2.6, the set

$$A = \Omega \setminus \limsup_{n\to+\infty} A_n$$

has full measure, $\mathbb{P}(A) = 1$. By construction, for any $\omega \in A$ there exists $N(\omega)$ such that

$$\max_{1\le k\le 2^n} \left| X_{\frac{k}{2^n}}(\omega) - X_{\frac{k-1}{2^n}}(\omega) \right| \le 2^{-\gamma n}, \qquad \forall n \ge N(\omega).$$

We claim that for every $\omega \in A$ the restriction of the function $t \to X_t(\omega)$ to $\mathscr{D}$ is γ–Hölder continuous, i.e.,

$$\exists\, C > 0 \quad \text{such that} \quad |X_t(\omega) - X_s(\omega)| \le C|t-s|^\gamma \tag{5.2.1}$$

for all $t, s \in \mathscr{D}$. Indeed, it is enough prove that (5.2.1) holds for $t, s \in \mathscr{D}$ with $|t-s| \le 2^{-N(\omega)}$.

Fixed $t, s \in \mathscr{D}$ such that $|t-s| \le 2^{-N(\omega)}$, there exists a unique $n \ge N(\omega)$ such that $2^{-n-1} < |t-s| \le 2^{-n}$. We consider the sequences $s_k \le s$, $t_k \le t$, $s_k, t_k \in \mathscr{D}$, defined by

$$s_k = \sum_{i=0}^{k} \frac{[2^i s]}{2^i}, t_k = \sum_{i=0}^{k} \frac{[2^i t]}{2^i},$$

where $[x]$ is the integer part of x. Such sequences are monotone increasing, and since $t, s \in \mathscr{D}$, they are eventually constant. Moreover,

$$s_{k+1} - s_k \le \frac{1}{2^{k+1}}, \quad t_{k+1} - t_k = \frac{1}{2^{k+1}}, \quad k \in \mathbb{N}.$$

Then,

$$X_t(\omega) - X_s(\omega) = X_{t_n}(\omega) - X_{s_n}(\omega) + \sum_{k\ge n}(X_{t_{k+1}}(\omega) - X_{t_k}(\omega)) - \sum_{k\ge n}(X_{s_{k+1}}(\omega) - X_{s_k}(\omega))$$

where the series are indeed finite sums. Hence

$$|X_t(\omega) - X_s(\omega)| \le 2^{-\gamma n} + 2\sum_{k\ge n} 2^{-\gamma(k+1)} = \frac{2^{-\gamma n}}{1-2^{-\gamma}} = \frac{2^{-\gamma}}{1-2^{-\gamma}}|t-s|^\gamma.$$

So (5.2.1) holds with $C = \frac{2^{-\gamma}}{1-2^{-\gamma}}$, for $t, s \in \mathscr{D}$ with $|t-s| \le 2^{-N(\omega)}$. Covering $[0,1]$ by a finite number of intervals with length $2^{-N(\omega)}$, we obtain that (5.2.1) holds for every $t, s \in \mathscr{D}$ (possibly, with a larger constant C). In particular, the mapping $t \mapsto X_t(\omega)$ is uniformly continuous on the dense set $\mathscr{D}$; therefore it admits a unique continuous extension to the whole $[0,1]$ which is what we need to define $\widetilde{X}_t(\omega)$.

Let us define for $\omega \in A$

$$\widetilde{X}_t(\omega) = \lim_{\mathscr{D}\ni s\to t} X_s(\omega),$$

and for $\omega \notin A$

$$\widetilde{X}_t(\omega) = 0.$$

It is clear that $\mathbb{P}(\{X_t \neq \widetilde{X}_t\}) = 0$ if $t \in \mathscr{D}$. For an arbitrary $t \in [0,1]$, there exists a sequence (t_h) in $\mathscr{D}$ such that X_{t_h} converges to $\widetilde{X}_t$ $\mathbb{P}$–a.e.. We use Egoroff's Theorem, see for instance [D, Theorem 7.5.1]: for any $\varepsilon > 0$ there exists $E_\varepsilon \in \mathscr{F}$ such that $\mathbb{P}(E_\varepsilon) < \varepsilon$ and X_{t_h} converges uniformly to $\widetilde{X}_t$ on $\Omega \setminus E_\varepsilon$. This implies convergence in measure, i.e. for any $\lambda > 0$

$$\lim_{h\to+\infty} \mathbb{P}(\{|X_{t_h} - \widetilde{X}_t| > \lambda\}) = 0.$$

On the other hand, we know that

$$\mathbb{P}(\{|X_{t_h} - X_t| > \lambda\}) \leq \frac{1}{\lambda^\beta}\mathbb{E}[|X_{t_h} - X_t|^\beta] \leq \frac{C|t_h - t|^{1+\alpha}}{\lambda^\beta}$$

and then

$$\lim_{h\to+\infty} \mathbb{P}(\{|X_{t_h} - X_t| > \lambda\}) = 0.$$

We deduce that $\widetilde{X}_t = X_t$ $\mathbb{P}$–a.e., see Exercise 5.6. Hence $(\widetilde{X}_t)$ is a version of (X_t). $\square$

Our aim now is to define a Borel measure (the Wiener measure) on the Banach space $C([0,1])$ endowed as usual with the sup norm. To do this we use the Brownian motion $(B_t)_{t\in[0,1]}$ on $(\mathbb{R}^{[0,1]}, \mathscr{F})$.

Lemma 5.2.8. *Let $\mathbb{P}^W$ be the Wiener measure on $(\mathbb{R}^{[0,1]}, \mathscr{F})$ and let $(B_t)_{t\in[0,1]}$ be the Brownian motion defined in Proposition 5.2.5. Then, for any $k \in \mathbb{N}$*

$$\mathbb{E}[(B_t - B_s)^k] = \begin{cases} 0 & \text{if } k \text{ odd} \\ \\ \frac{k!}{(\frac{k}{2})!2^{\frac{k}{2}}}|t-s|^{\frac{k}{2}} & \text{if } k \text{ even.} \end{cases}$$

Proof. Let us take $0 < s < t$. Since the law of $B_t - B_s$ is $\mathscr{N}(0, t-s)$, we get

$$I_k := \mathbb{E}[(B_t - B_s)^k] = \frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}} x^k e^{-\frac{x^2}{2(t-s)}}\,dx.$$

As a consequence $I_k = 0$ if k is odd, whereas integrating by parts one obtains $I_0 = 1$, $I_2 = (t-s)$ and $I_{2h} = (2h-1)(t-s)I_{2h-2}$ for $h \geq 2$. $\square$

Lemma 5.2.8 and Theorem 5.2.7 yield that, fixed any $\gamma < 1/2$, there exists a version $(\widetilde{B}_t)$ of (B_t) such that the trajectories $t \mapsto \widetilde{B}_t(\omega)$ are γ–Hölder continuous, in particular they are continuous. For any t, the random variables B_t and $\widetilde{B}_t$ have the same law, $\mathbb{P}^W \circ B_t^{-1} = \mathbb{P}^W \circ \widetilde{B}_t^{-1}$. The map $P : \mathbb{R}^{[0,1]} \to \mathbb{R}^n$,

$$P(\omega) = (\widetilde{B}_{t_1}(\omega), \ldots, \widetilde{B}_{t_n}(\omega)) \tag{5.2.2}$$

is measurable for any choice $t_1, \ldots, t_n \in [0,1]$, and the image measure of $\mathbb{P}^W$ under the map P is the same as the image measure of $\mathbb{P}^W$ under the map

$$\omega \mapsto (B_{t_1}(\omega), \ldots, B_{t_n}(\omega)).$$

Finally, for $C \in \mathscr{C}_F$, $\mathbb{P}^W \circ P^{-1}(E) = \mu_F(E)$. We leave the verification of these properties as an exercise, see Exercise 5.5.

We recall some facts. The first one is the characterisation of the dual space $(C([0,1]))^*$. We denote by $\mathscr{M}([0,1])$ the space of all real finite measures on $[0,1]$; it is a real Banach space with the norm $\|\mu\| = |\mu|([0,1])$, see Exercise 5.4.

Theorem 5.2.9 (Riesz representation Theorem)**.** *There is a linear isometry between the space $\mathscr{M}([0,1])$ of finite measures and $(C([0,1]))^*$, i.e. $L \in (C([0,1]))^*$ iff there exists $\mu \in \mathscr{M}([0,1])$ such that*

$$L(f) = \int_{[0.1]} \omega(t)\mu(dt), \qquad \forall \omega \in C([0,1]).$$

In addition $\|L\| = |\mu|([0,1])$.

We refer to [D, Theorem 7.4.1] for a proof.

We define the σ–algebra $\mathscr{C}'_F$ for $F = \{t_1, \dots, t_n\}$ as the family of sets

$$C = \{\omega \in C([0,1]) : (\omega(t_1), \dots, \omega(t_n)) \in B\},$$

where $B \in \mathscr{B}(\mathbb{R}^n)$. We also define the algebra

$$\mathscr{A}' = \bigcup_{F \subset [0,1],\ F \text{ finite}} \mathscr{C}'_F$$

and we denote by $\mathscr{F}'$ the σ–algebra generated by $\mathscr{A}'$. Using Theorem 5.2.9 and the fact that any Dirac measure δ_t is in $(C([0,1]))^*$, it is clear that $\mathscr{F}' \subset \mathscr{B}(C([0,1]))$; indeed, if $F = \{t_1, \dots, t_n\}$ and $B \in \mathscr{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} C :=&\{\omega \in C([0,1]) : (\omega(t_1), \dots, \omega(t_n)) \in B\} \\ =&\{\omega \in C([0,1]) : (\delta_{t_1}(\omega), \dots, \delta_{t_n}(\omega) \in B\} \in \mathscr{E}(C([0,1]), \{\delta_{t_1}, \dots, \delta_{t_n}\}). \end{aligned}$$

We have also the reverse inclusion, i.e. $\mathscr{B}(C([0,1])) \subset \mathscr{F}'$; the proof is similar to the proof of Theorem 2.1.1. Indeed, fix $\omega_0 \in C([0,1])$, $r > 0$ and let $\mathscr{D}$ be the set in the proof of the Kolmogorov continuity Theorem 5.2.7. Note that $\omega \in \overline{B}(\omega_0, r)$ if and only if $\|\omega - \omega_0\|_\infty \le r$, and by continuity this is equivalent to $|\omega(t) - \omega_0(t)| \le r$ for any $t \in \mathscr{D}$. Then

$$\overline{B}(\omega_0, r) = \bigcap_{n \in \mathbb{N}} \left\{ \omega \in C([0,1]) : \omega\Big(\frac{k}{2^n}\Big) \in \Big[r - \omega_0\Big(\frac{k}{2^n}\Big), r + \omega_0\Big(\frac{k}{2^n}\Big)\Big],\ \forall k = 0, \dots, 2^n \right\}.$$

The set in the right hand side belongs to $\mathscr{F}'$. Since $C([0,1])$ is separable then as in the proof of Theorem 2.1.1 $\mathscr{B}(C([0,1])) \subset \mathscr{F}'$.

Now we consider a version $(\widetilde{B}_t)$ of (B_t), constructed using the Kolmogorov continuity Theorem 5.2.7 and a set $A \in \mathscr{F}$ such that $\mathbb{P}^W(A) = 1$ and $t \mapsto \widetilde{B}_t(\omega)$ is continuous for any $\omega \in A$. We define the restricted σ–algebra

$$\mathscr{F}_A = \{E \cap A : E \in \mathscr{F}\}$$

and the restriction $\mathbb{P}^W_A$ of $\mathbb{P}^W$ to $\mathscr{F}_A$.

Proposition 5.2.10. *The map* $\widetilde{B} : (A, \mathscr{F}_A) \to (C([0,1]), \mathscr{B}(C([0,1])))$ *defined by*

$$\widetilde{B}(\omega)(t) := \widetilde{B}_t(\omega)$$

is measurable. The image measure $\mathbb{P}_A^W \circ \widetilde{B}^{-1}$*, called the* Wiener measure on $C([0,1])$*, has the property that for any* $E \in \mathscr{C}'_F$

$$\mathbb{P}_A^W \circ \widetilde{B}^{-1}(E) = \mu_F(\widetilde{B}^{-1}(E)). \tag{5.2.3}$$

Proof. We know that $\mathscr{B}(C([0,1])) = \mathscr{F}'$, so it is sufficient to prove that for every finite set $F = \{t_1, \dots, t_n\} \subset [0,1]$, we have

$$\widetilde{B}^{-1}(C) \in \mathscr{F}_A, \quad \forall C \in \mathscr{C}'_F.$$

Let $E \in \mathscr{B}(\mathbb{R}^n)$ and

$$C = \{\omega \in A : (\omega(t_1), \dots, \omega(t_n)) \in E\}.$$

Then

$$\widetilde{B}^{-1}(C) = \{\omega \in A : (\widetilde{B}_{t_1}(\omega), \dots, \widetilde{B}_{t_n}(\omega)) \in E\} = A \cap P^{-1}(E)$$

where P is the map defined in (5.2.2). Since P is measurable, then $\widetilde{B}^{-1}(C) \in \mathscr{F}_A$ The last assertion follows from the fact that $\mathbb{P}^W(A) = 1$ and

$$(\mathbb{P}_A^W \circ \widetilde{B}^{-1})(C) = \mathbb{P}^W(A \cap P^{-1}(E)) = \mathbb{P}^W(P^{-1}(E)) = \mu_F(P^{-1}(E))$$

since $P^{-1}(E) \in \mathscr{C}_F$. □

5.3 Exercises

Exercise 5.1. Use Corollary 5.1.4 to prove that two random variables X and Y on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ are independent if and only if $\mathbb{P} \circ (X, Y)^{-1} = (\mathbb{P} \circ X^{-1}) \otimes (\mathbb{P} \circ Y^{-1})$.

Exercise 5.2. Prove Proposition 5.2.1.

Exercise 5.3. Prove the *Chebychev's inequality*: if $\beta > 0$ and f is a measurable function uch that $|f|^\beta \in L^1(\Omega, \mu)$, we have for any $\lambda > 0$

$$\mu(\{|f| \geq \lambda\}) \leq \frac{1}{\lambda^\beta} \int_\Omega |f|^\beta d\mu.$$

Exercise 5.4. Prove that the vector space $\mathscr{M}([0,1])$ with norm given by the total variation is a Banach space.

Exercise 5.5. Prove that the map $P : \mathbb{R}^{[0,1]} \to \mathbb{R}^n$, defined by (5.2.2), is measurable. Prove that $\mathbb{P}^W \circ P^{-1} = \mathbb{P}^W \circ T^{-1}$ with

$$T(\omega) = (\omega(t_1), \dots, \omega(t_n)) = (B_{t_1}(\omega), \dots, B_{t_n}(\omega)).$$

Prove in addition that if $C \in \mathscr{C}_F$, then

$$(\mathbb{P}^W \circ P^{-1})(C) = \mu_F(C).$$

Exercise 5.6. Let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, that converge in law to X and to Y. Prove that $X = Y$, $\mathbb{P}$–a.e.

Bibliography

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