Solutions to the exercises of Lecture 5 of the ISem 2015/16

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Exercise 5.1
Claim: The real valued random variables X and Y on a probability space $(\Omega, \mathscr{F}, P)$ are independent if and only if the joint distribution $P \circ (X,Y)^{-1}$ coincides with $(P \circ X^{-1}) \otimes (P \circ Y^{-1})$.
Proof: 1) By Corollary 5.1.4, the independence of X and Y implies

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)]$$

for all bounded and measurable functions f and g. In particular, it we choose $f = \mathbf{1}_{B_1}$ and $g = \mathbf{1}_{B_2}$ for Borel sets $B_1, B_2 \subset \mathbb{R}$, we obtain

$$\begin{aligned}(P \circ (X,Y)^{-1})(B_1 \times B_2) &= P((X,Y) \in B_1 \times B_2) \\ &= \int_{\mathbb{R}^2} \mathbf{1}_{B_1}(x)\mathbf{1}_{B_2}(y) d(P \circ (X,Y)^{-1})(x,y) \\ &= E[\mathbf{1}_{B_1}(X)\mathbf{1}_{B_2}(Y)] = E[\mathbf{1}_{B_1}(X)]E[\mathbf{1}_{B_2}(Y)] \\ &= P(X \in B_1)P(Y \in B_2) = ((P \circ X^{-1}) \otimes (P \circ Y^{-1}))(B_1 \times B_2).\end{aligned}$$

Thus $P \circ (X,Y)^{-1}$ coincides with $P \circ X^{-1} \otimes P \circ Y^{-1}$ on cylindrical sets. Since they are stable under intersections and generate the Borel σ-algebra on $\mathbb{R}^2$, we have $(P \circ (X,Y)^{-1}) = ((P \circ X^{-1}) \otimes (P \circ Y^{-1}))$.
2) We now assume $(P \circ (X,Y)^{-1}) = ((P \circ X^{-1}) \otimes (P \circ Y^{-1}))$. Then one has

$$\begin{aligned}P(X \in B_1, Y \in B_2) &= (P \circ (X,Y)^{-1})(B_1 \times B_2) = ((P \circ X^{-1}) \otimes (P \circ Y^{-1}))(B_1 \times B_2) \\ &= P(X \in B_1)P(Y \in B_2)\end{aligned}$$

for Borel sets $B_1, B_2 \subset \mathbb{R}$, which is the desired result.

Exercise 5.2
Assumptions: Let μ be a real valued finitely additive set function on an algebra $\mathscr{A}$.
Claim: μ is countably additive (σ-additive) if and only if it is continuous at $\emptyset$, i. e.,

$$\lim_{n \to +\infty} \mu(A_n) = 0$$

for every decreasing sequence of sets $(A_n) \subset \mathscr{A}$ with $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.
Proof: 1) Let first be μ continuous at $\emptyset$. Let $(A_n) \subset \mathscr{A}$ be a sequence of pairwise disjoint sets. Define $B_k := \bigcup_{n \geq k} A_n$ for all $k \in \mathbb{N}$. From $\bigcap_{k \in \mathbb{N}} B_k = \emptyset$ and the fact that (B_k) is decreasing, we infer $\lim_{k \to +\infty} \mu(B_k) = 0$. The finite additivity thus implies

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \mu\Big(A_1 \cup \ldots \cup A_{k-1} \cup \bigcup_{n \geq k} A_n\Big) = \mu(A_1 \cup \ldots \cup A_{k-1} \cup B_k)$$

$$= \sum_{n=1}^{k-1} \mu(A_n) + \mu(B_k)$$

for all $k \in \mathbb{N}$. So we derive

$$\lim_{k \to +\infty} \left| \mu\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) - \sum_{n=1}^{k-1} \mu(A_n) \right| = \lim_{k \to +\infty} |\mu(B_k)| = 0,$$

which yields (the existence of)

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) = \sum_{l=1}^{\infty} \mu(A_l).$$

2) Conversely, let μ be countably additive. Let $(A_n) \subset \mathscr{A}$ be a decreasing sequence with $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$. We obtain

$$\mu(A_n) = \mu\Big(\bigcup_{k \geq n} (A_k \setminus A_{k+1}) \Big) = \sum_{k=n}^{\infty} \mu(A_k \setminus A_{k+1})$$

for all $n \in \mathbb{N}$, using that the sets $A_k \setminus A_{k+1}$ are disjoint. Since $\mu(A_n)$ is finite for each $n \in \mathbb{N}$, for all $\varepsilon > 0$ there exists an index $n_0 \in \mathbb{N}$ such that

$$\mu(A_l) = \sum_{k=l}^{\infty} \mu(A_k \setminus A_{k-1}) < \varepsilon$$

for every $l \geq n_0$, which is the desired continuity of μ at $\emptyset$.

Exercise 5.3
Assumption: Let μ be a positive measure on $(\Omega, \mathcal{A})$.
Claim: Given a measurable function f with $|f|^\beta \in L^1(\Omega, \mu)$ for some $\beta > 0$, one has

$$\mu(\{|f| \geq \lambda\}) \leq \lambda^{-\beta} \int_\Omega |f|^\beta \,\mathrm{d}\mu$$

for all $\lambda > 0$.
Proof: The monotonicity of the integral yields

$$\int_\Omega |f|^\beta \,\mathrm{d}\mu = \int_{\{|f| \geq \lambda\}} |f|^\beta \,\mathrm{d}\mu + \int_{\{|f| < \lambda\}} |f|^\beta \,\mathrm{d}\mu \geq \int_{\{|f| \geq \lambda\}} \lambda^\beta \,\mathrm{d}\mu = \lambda^\beta \mu(\{|f| \geq \lambda\}).$$

Exercise 5.4
Assumptions: Let $\mathscr{M}([0,1])$ be the space of all real finite measures on $[0,1]$.
Claim: $\mathscr{M}([0,1])$ together with the norm $\|\mu\| = |\mu|\,([0,1])$ is a Banach space.
Proof: It is clear that the set of all real finite measures on $[0,1]$ is a vector space. In the following we use

$$\|\mu\| = |\mu|\,([0,1]) = \sup \left\{ \int_{[0,1]} f \,\mathrm{d}\mu : f \in C([0,1]), \|f\|_\infty \leq 1 \right\},$$

compare equation (1.1.3) in Lecture 1. We observe that we can also write

$$\|\mu\| = \sup \left\{ \left| \int_{[0,1]} f \,\mathrm{d}\mu \right| : f \in C([0,1]), \|f\|_\infty \leq 1 \right\}.$$

We now show that $\|\cdot\|$ is a norm on $\mathscr{M}([0,1])$. Let $\mu, \nu \in \mathscr{M}([0,1])$. Then

$$\begin{aligned} \|\mu+\nu\| &= \sup\left\{\left|\int_{[0,1]} f \,\mathrm{d}(\mu+\nu)\right| : f \in C([0,1]), \|f\|_\infty \le 1\right\} \\ &\le \sup\left\{\left|\int_{[0,1]} f \,\mathrm{d}\mu\right| + \left|\int_{[0,1]} f \,\mathrm{d}\nu\right| : f \in C_b([0,1]), \|f\|_\infty \le 1\right\} \\ &\le \sup\left\{\left|\int_{[0,1]} f \,\mathrm{d}\mu\right| : f \in C([0,1]), \|f\|_\infty \le 1\right\} \\ &\quad + \sup\left\{\left|\int_{[0,1]} f \,\mathrm{d}\nu\right| : f \in C([0,1]), \|f\|_\infty \le 1\right\} \\ &= \|\mu\| + \|\nu\|. \end{aligned}$$

For $\mu \in \mathscr{M}([0,1])$ and $\alpha \in \mathbb{R}$ we compute

$$\begin{aligned} \|\alpha\mu\| &= \sup\left\{\left|\int_{[0,1]} f \,\mathrm{d}(\alpha\mu)\right| : f \in C([0,1]), \|f\|_\infty \le 1\right\} \\ &= \sup\left\{|\alpha|\left|\int_{[0,1]} f \,\mathrm{d}\mu\right| : f \in C([0,1]), \|f\|_\infty \le 1\right\} = |\alpha| \, \|\mu\|. \end{aligned}$$

For all $\mu \in \mathscr{M}([0,1])$ we have

$$\|\mu\| = 0 \iff \int_{[0,1]} f \,\mathrm{d}\mu = 0 \text{ for all } f \in C([0,1]) \text{ with } \|f\|_\infty \le 1 \iff \mu = 0.$$

So we have seen that $\|\cdot\|$ is a norm on $\mathscr{M}([0,1])$.
It remains to show the completeness. Let $(\mu_n)_n \subset M([0,1])$ a Cauchy sequence, i.e.,

$$|\mu_n - \mu_m|([0,1]) := \sup\left\{\sum_{h=1}^{\infty} |\mu_n(E_h) - \mu_m(E_h)| : \; E_h \in \mathcal{B}([0,1]) \text{ pairwise disjoint}\right\} \xrightarrow{m,n\to\infty} 0.$$

In particular we have

$$\sup_{A\in\mathcal{B}([0,1])} |\mu_n(A) - \mu_m(A)| \xrightarrow{m,n\to\infty} 0. \tag{1}$$

Therefore we can define $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ for all $A \in \mathcal{B}([0,1])$. Moreover, (1) implies

$$\sup_{A\in\mathcal{B}([0,1])} |\mu_n(A) - \mu(A)| \xrightarrow{m,n\to\infty} 0. \tag{2}$$

Clearly, μ is a finite set function on $\mathcal{B}([0,1])$, which satisfies $\mu(\emptyset) = 0$. It is straight forward to check that μ is finitely additive. So, it remains to prove that μ is σ-additive, or, equivalently by Exercise 5.2, to show that μ is continuous at $\emptyset$. Let $\varepsilon > 0$ and $(A_k)_k \subset \mathcal{B}([0,1])$ be a decreasing sequence with $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$. By (2) we fix an index $n_0 \in \mathbb{N}$ such that

$$\sup_{A\in\mathcal{B}([0,1])} |\mu_n(A) - \mu(A)| < \varepsilon$$

for all $n \ge n_0$. Then the continuity of μ_n at $\emptyset$ yields

$$|\mu(A_k)| \le |\mu(A_k) - \mu_n(A_k)| + |\mu_n(A_k)| \le \varepsilon + |\mu_n(A_k)| \xrightarrow{k\to\infty} \varepsilon,$$

for all $n \geq n_0$.

Exercise 5.5
Assumptions: Let $t_1, \dots, t_n \in [0,1]$. Define the map $P : \mathbb{R}^{[0,1]} \to \mathbb{R}^n$ via

$$P(\omega) := \left(\tilde{B}_{t_1}(\omega), \dots, \tilde{B}_{t_n}(\omega)\right).$$

Claim 1: P is $\mathcal{F}$-measurable.
Proof: By the proof of Theorem 5.2.7, there is an $\mathcal{F}$-measurable set $A \subset \mathbb{R}^{[0,1]}$ of full measure with

$$\tilde{B}_t(\omega) = \lim_{\mathcal{D} \ni s \to t} B_s(\omega) = \lim_{\mathcal{D} \ni s \to t} \omega(s).$$

for all $\omega \in A$ and $t \in [0,1]$. Moreover, $\widetilde{B}_t(\omega) = 0$ for $\omega \in A^c$ and $t \in [0,1]$. We can choose sequences $(s_1^m)_{m \in \mathbb{N}}, \dots, (s_n^m)_{m \in \mathbb{N}}$ with

$$P(\omega) = \lim_{m \to \infty} (\omega(s_1^m), \dots, \omega(s_n^m)) = \lim_{m \to \infty} P_{\{s_1^m, \dots, s_n^m\}}(\omega).$$

for $\omega \in A$. The maps $P_{\{s_1^m, \dots, s_n^m\}}$ for $m \in \mathbb{N}$ are $\mathcal{F}$-measurable by the definition of the σ-algebra $\mathcal{F}$, so that the restriction $P : A \to \mathbb{R}$ is measurable. Since $P = 0$ on A^c, the claim follows thanks to the pointwise convergence.

Claim 2: One has

$$\mathbb{P}^W \circ P^{-1} = \mathbb{P}^W \circ T^{-1},$$

where $\mathbb{P}^W \circ T^{-1}$ is the law of the random variable defined by

$$T(\omega) = (\omega(t_1), \dots, \omega(t_n)) = (B_{t_1}(\omega), \dots, B_{t_n}(\omega))$$

for $\omega \in \mathbb{R}^{[0,1]}$.
Proof: Theorem 5.2.7 yields that for each $t \in [0,1]$ there is a null set A_t with $B_t(\omega) = \tilde{B}_t(\omega)$ for all $\omega \notin A_t$. Defining the null set N by $N := \bigcup_{j=1}^n A_{t_j}$, we obtain $T(\omega) = P(\omega)$ for all $\omega \notin N$ and hence $P = T$ almost surely. For a Borel set $B \subset \mathbb{R}^n$ we conclude

$$\begin{aligned}
(\mathbb{P}^W \circ P^{-1})(B) &= \int_{\mathbb{R}^{[0,1]}} \mathbf{1}_B(P(\omega)) \, \mathrm{d}\, \mathbb{P}^W(\omega) = \int_A \mathbf{1}_B(P(\omega)) \, \mathrm{d}\, \mathbb{P}^W(\omega) \\
&= \int_A \mathbf{1}_B(T(\omega)) \, \mathrm{d}\, \mathbb{P}^W(\omega) = \int_{\mathbb{R}^{[0,1]}} \mathbf{1}_B(T(\omega)) \, \mathrm{d}\, \mathbb{P}^W(\omega) \\
&= (\mathbb{P}^W \circ T^{-1})(B),
\end{aligned}$$

which implies the assertion.

Claim 3: Let $F = \{t_1, \dots, t_n\}$, $C \in \mathscr{C}_F$ and $B \subset \mathbb{R}^n$ be a Borel set with $C = P_F^{-1}(B)$. Then we have

$$\mu_F(C) = (\mathbb{P}^W \circ P^{-1})(B).$$

Proof: We have

$$C = \{\omega \in \mathbb{R}^{[0,1]} : (\omega(t_1), \dots, \omega(t_n)) \in B\} = \{\omega \in \mathbb{R}^{[0,1]} : T(\omega) \in B\}.$$

Using the Theorem 5.2.3. and Claim 2 we conclude

$$\mu_F(C) = \mathbb{P}^W(C) = \mathbb{P}^W(T \in B) = (\mathbb{P}^W \circ T^{-1})(B) = (\mathbb{P}^W \circ P^{-1})(B).$$

Exercise 5.6

Assumption: Let $(X_n)_n$ a sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$ converging to the random variables X and Y on $(\Omega, \mathscr{F}, P)$ in law.

Claim: In general the statement $X = Y$ a. s. is not true.

Proof: Consider a sequence $(X_n)_n$ of random variables with distribution $N(0,1)$. Clearly by definition of convergence in law it converges to any $N(0,1)$ distributed X since $P \circ X^{-1} = P \circ X_n^{-1}$ for all $n \in \mathbb{N}$. Unfortunately $Y = -X$ is also $N(0,1)$ distributed and we have $P \circ Y^{-1} = P \circ X_n^{-1}$ for all $n \in \mathbb{N}$. But $\{Y = X\} = \{X = 0\}$ is a set of measure zero.

Exercise 5.6, second try

Assumption: Let $(X_n)_n$ a sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$ converging to the random variables X and Y on $(\Omega, \mathcal{A}, P)$ in probability.

Claim: $X = Y$ almost surely.

Proof: Given $\varepsilon > 0$ we have $P(|X - Y| > \varepsilon) = 0$. Indeed the triangle inequality yields

$$\begin{aligned} P(|X - Y| > \varepsilon) &\leq P(|X - X_n| + |X_n - Y| > \varepsilon) \\ &\leq P(|X - X_n| > \tfrac{\varepsilon}{2} \text{ or } |X_n - Y| > \tfrac{\varepsilon}{2}) \\ &\leq P(|X - X_n| > \tfrac{\varepsilon}{2}) + P(|Y - X_n| > \tfrac{\varepsilon}{2}) \xrightarrow{n \to \infty} 0. \end{aligned}$$

Moreover, exploiting the continuity of the measure P we obtain

$$P(X \neq Y) = P\Big(\bigcup_{n=1}^{\infty} \{|X - Y| > \tfrac{1}{n}\}\Big) = \lim_{n \to \infty} P(|X - Y| > \tfrac{1}{n}) = 0,$$

which implies the claimed result.