Exercise 4.1

(i) Prove that the dual space of $\mathbb{R}^\infty$ is $\mathbb{R}^\infty_c$.

(ii) Prove that the closure of $\mathbb{R}^\infty_c$ in $\mathbb{R}^\infty$ with respect to the topology induced by the norm $\|\cdot\|_{\ell^2}$ is ℓ^2.

Proof of (i): We show equality as sets.

"$\mathbb{R}^\infty_c \subset (\mathbb{R}^\infty)^*$": Let $(\xi_k) \in \mathbb{R}^\infty_c$. Then

$$f_\xi \colon \mathbb{R}^\infty \to \mathbb{R}$$
$$f_\xi(x) := \sum_{k=1}^{\infty} \xi_k x_k$$

defines a linear function. Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^\infty$ such that

$$x^n \xrightarrow{\mathbb{R}^\infty} x \in \mathbb{R}^\infty \text{ i.e. } x^n_k \to x_k \quad \forall k \in \mathbb{N}.$$

Since there is a number $N \in \mathbb{N}$ such that $\xi_j = 0 \; \forall j > N$, the following holds:

$$f_\xi(x^n) = \sum_{k=1}^{N} \xi_k x^n_k \overset{n\to\infty}{\longrightarrow} \sum_{k=1}^{N} \xi_k x_k = f_\xi(x),$$

i.e. f_ξ is continuous. Thus $(\xi_k) \cong f_\xi \in (\mathbb{R}^\infty)^*$.

"$\mathbb{R}^\infty_c \supset (\mathbb{R}^\infty)^*$": Let $f \in (\mathbb{R}^\infty)^*$ and define $\xi_k := f(e_k) \; \forall k \in \mathbb{N}$. Suppose for contradiction that $(\xi_k) \notin \mathbb{R}^\infty_c$, i.e. $\xi_{k_n} \neq 0 \; \forall n \in \mathbb{N}, k_n \uparrow \infty$. Define $x^n := e_{k_n} \xi_{k_n}^{-1} \; \forall n \in \mathbb{N}$. Then $\mathbb{R}^\infty_c \supset (x^n) \to 0$ because $d(x^n, 0) \leq \frac{1}{2^{k_n}} \to 0$ as $n \to \infty$, but $f(x^n) = 1 \not\to 0$. Hence f is not continuous (in 0), which is a contradiction to $f \in (\mathbb{R}^\infty)^*$. Consequently $(\xi_k) \in \mathbb{R}^\infty_c$ and $f = f_\xi$. □

Proof of (ii): The formulation of this exercise is somewhat unfortunate in that it is up to interpretation what the topology on $\mathbb{R}^\infty$ induced by the ℓ^2-norm is supposed to be. As suggested by Jürgen Voigt, we show that the map $E \colon x \mapsto x$ is continuous as map $(\ell^2, \|\cdot\|_{\ell^2}) \to (\mathbb{R}^\infty, d)$. It is then clear that E is the unique continuous extension of its restriction to $\mathbb{R}^\infty_c$ as $\mathbb{R}^\infty_c$ is dense in ℓ^2. In fact, for the latter let $a \in \ell^2$ and define $x^n := (a_1, a_2, \ldots, a_n, 0, \ldots) \in \mathbb{R}^\infty_c$. Then $x^n \to a$ in ℓ^2 as $n \to \infty$ since $\|a - x^n\|_{\ell^2} = \sum_{j=n+1}^{\infty} a_j^2$ and for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\sum_{j=n}^{\infty} a_j^2 < \varepsilon \; \forall n > N$.

It remains to show that E is continuous in the above sense. So suppose that $x^n \to x$ in ℓ^2. Then $x^n_k \to x_k$ for all $k \in \mathbb{N}$, which is equivalent to $d(x^n, x) \to 0$ as $n \to \infty$. So if $x^n \to x$ in ℓ^2, then $Ex^n \to Ex$ in $(\mathbb{R}^\infty, d)$ and E is therefore continuous. □

Exercise 4.2

Let X be a real Hilbert space, and let $B\colon X \times X \to \mathbb{R}$ be bilinear, symmetric and continuous. Prove that there exists a unique self-adjoint operator $Q \in \mathcal{L}(X)$ such that $B(x, y) = \langle Qx, y\rangle_X$ for every $x, y \in X$.

Proof. Fix $x \in X$. Then $B(x, \cdot) \in X^*$ and by Riesz–Fréchet's theorem we get

$$\exists!\ Qx \in X \text{ such that } B(x, y) = \langle Qx, y\rangle_X \ \forall y \in X, \|B(x, \cdot)\| = \|Qx\|.$$

Moreover, as $|B(x, y)| \leq C\|x\|\|y\|$, which is a consequence of the uniform boundedness principle, it follows that

$$\|Qx\| = \sup_{\|y\|\leq 1} |B(x, y)| \leq C\|x\|.$$

So Q is a bounded operator. Since B is symmetric and X is a real Hilbert space we get

$$\langle Qx, y\rangle = B(x, y) = B(y, x) = \langle Qy, x\rangle = \langle y, Q^*x\rangle = \langle Q^*x, y\rangle$$

for all $y \in X$. Hence $Q^* = Q$ and Q is self-adjoint. □

Exercise 4.3

Let $L\colon \ell^2 \to \ell^2$ be the operator defined by $Lx = (x_2, x_1, x_4, x_3, \ldots)$, for $x = (x_1, x_2, x_3, x_4, \ldots) \in \ell^2$. Show that L is self-adjoint and $\langle Le_k, e_k\rangle_{\ell^2} = 0$ and that L is not compact.

Proof. L self-adjoint: For $x, y \in \ell^2$ we have $\langle Lx, y\rangle_{\ell^2} \leq \|Lx\|_{\ell^2}\|y\|_{\ell^2} = \|x\|_{\ell^2}\|y\|_{\ell^2}$ and

$$\begin{aligned}\langle Lx, y\rangle_{\ell^2} &= x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 + \ldots \\ &= x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 + \ldots = \langle x, Ly\rangle_{\ell^2}\end{aligned}$$

So L is bounded and self-adjoint.

L zero on diagonal: We know that $\langle e_k, e_j\rangle_{\ell^2} = \delta_{kj}$. As

$$\langle Le_k, e_k\rangle_{\ell^2} = \begin{cases} \langle e_{k+1}, e_k\rangle_{\ell^2} & k \text{ odd,} \\ \langle e_{k-1}, e_k\rangle_{\ell^2} & k \text{ even,}\end{cases}$$

it follows that $\langle Le_k, e_k\rangle_{\ell^2} = 0\ \forall k \in \mathbb{N}$.

L not compact: Define $U_1 := \{x \in \ell^2 : \|x\| \le 1\}$. Since $\|x\| = \|Lx\|$, the image of U_1 under L is again U_1. Thus L is not compact as ℓ^2 is infinite-dimensional.

Note that this is no contradiction to Proposition 4.2.2 because L is not nonnegative. $\square$

Exercise 4.4

Check the computation of the integral in (4.2.7).

Proof. To simplify notation we omit the index k in the subscripts. The following calculation is straightforward and only involves substitutions and completing the square. We note that $1 - 2\alpha\lambda > 0$ by assumption.

$$\begin{aligned}
\int_{\mathbb{R}} \exp(\alpha x^2)\exp(-\tfrac{1}{2\lambda}(x-a)^2)\,\mathrm{d}x &= \int_{\mathbb{R}} \exp\big(\tfrac{1}{2\lambda}(-(1-2\alpha\lambda)x^2 + 2ax - a^2)\big)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{1-2\alpha\lambda}} \exp(-\tfrac{a^2}{2\lambda}) \int_{\mathbb{R}} \exp\big(\tfrac{1}{2\lambda}(-u^2 + \tfrac{2a}{\sqrt{1-2\alpha\lambda}}u)\big)\,\mathrm{d}u \\
&= \frac{1}{\sqrt{1-2\alpha\lambda}} \exp\big(\tfrac{a^2}{2\lambda(1-2\alpha\lambda)} - \tfrac{a^2}{2\lambda}\big) \int_{\mathbb{R}} \exp\big(-\tfrac{1}{2\lambda}(u - \tfrac{a}{\sqrt{1-2\alpha\lambda}})^2\big)\,\mathrm{d}u \\
&= \frac{1}{\sqrt{1-2\alpha\lambda}} \exp\big(\tfrac{\alpha a^2}{1-2\alpha\lambda}\big) \int_{\mathbb{R}} \exp(-\tfrac{v^2}{2\lambda})\,\mathrm{d}v \\
&= \sqrt{2\pi\lambda}\frac{1}{\sqrt{1-2\alpha\lambda}} \exp\big(\tfrac{\alpha a^2}{1-2\alpha\lambda}\big)
\end{aligned}$$

This shows in particular that the integral is finite. However, in the current version of the lecture notes the formula given in (4.2.7) is missing a factor of $\sqrt{2\pi\lambda}$ that cancels with a corresponding factor in the previous displayed equation. $\square$

Exercise 4.5

Prove that if X is a separable Hilbert space, then $h \in R_\gamma(j(X^*))$ if and only if $h = Qx$, $x \in X$, and that in this case $|h|_H = \|Q^{1/2}x\|_X$.

Proof. For $\tilde{f} \in X^*$ let us write f for the element of X such that $\tilde{f}(x) = \langle f, x\rangle_X$ for all $x \in X$, and vice versa. Let $\tilde{f} \in X^*$ and suppose $h = R_\gamma(j(\tilde{f}))$ for $h \in X$. Then

$$\langle h, g\rangle = \tilde{g}(h) = \int_X \langle f, x-a\rangle_X \langle g, x-a\rangle_X \,\mathrm{d}\gamma(x) = \langle Qf, g\rangle$$

for all $g \in X$ by (4.2.4). So $h \in \operatorname{rg} Q$. The same reasoning applies the other way around.

If $h = Qf$, then $|h|_H = \|j(\tilde{f})\|_{L^2(X,\gamma)}$ by Proposition 3.1.2. But

$$\|j(\tilde{f})\|^2_{L^2(X,\gamma)} = \langle Qf, f\rangle_X = \|Q^{1/2} f\|^2_X,$$

which establishes the last claim. □

Exercise 4.6

Modify the proof of Theorem 4.2.6 in order to consider also degenerate Gaussian measures.

We have to slightly adapt the formulation of Theorem 4.2.6 to accommodate the case of general Gaussian measures. Note that $\gamma = \mathscr{N}(a, Q)$ is degenerate if and only if there exists $\tilde{f} \in X^* \setminus \{0\}$ such that $\gamma \circ \tilde{f}^{-1}$ is the Dirac measure $\delta_{\tilde{f}(a)}$, which is the case if and only if

$$B_\gamma(\tilde{f}, \tilde{f}) = \int_{\mathbb{R}} |t - \tilde{f}(a)|^2 \, \mathrm{d}(\gamma \circ \tilde{f}^{-1})(t) = 0.$$

So this is equivalent for an $f \in X \setminus \{0\}$ to exist such that $\langle Qf, f\rangle = 0$. As $\ker Q = \ker Q^{1/2}$, one has $\langle Qf, f\rangle = 0$ if and only if $f \in \ker Q$.

Theorem 4.2.6 (general). *Let $\gamma = \mathscr{N}(a, Q)$ be a Gaussian measure in a separable Hilbert space X. Let (λ_k) be the strictly positive eigenvalues of Q and (e_k) be the corresponding orthonormal system of eigenvectors in X. For all k and $x \in X$ we set $x_k := \langle x, e_k\rangle$. Then*

$$X^*_\gamma = \Big\{ f\colon X \to \mathbb{R} : f(x) = \sum_k (x_k - a_k) z_k \lambda_k^{-1/2} \text{ for a } z \in X \Big\}$$

and the Cameron–Martin space is the range of $Q^{1/2}$, i.e.,

$$H = \Big\{ x \in (\ker Q)^\perp : \sum_k x_k^2 \lambda_k^{-1} < \infty \Big\}.$$

For $h = Q^{1/2} z \in H$ we have

$$\hat{h}(x) = \sum_k (x_k - a_k) z_k \lambda_k^{-1/2}.$$

Proof. We use the notation from the proof in the lecture notes. We denote by P the orthogonal projection in X onto $(\ker Q)^\perp$. It follows in exactly the same way as in the lecture notes that

$$\|f\|^2_{L^2(X,\gamma)} = \|Pz\|^2_X.$$

No further change is needed for the proof of $V \subset X^*_\gamma$. Also the opposite inclusion follows as before after noting that $z^{(n)} - z$ is in the closure of the range of $Q^{1/2}$, which is equal to $(\ker Q)^\perp$. So $\|P(z^{(n)} - z)\|^2_X = \|z^{(n)} - z\|^2_X$.

Regarding the Cameron–Martin space, let $f \in X^*_\gamma$ be given, i.e., $f(x) = \sum_k (x_k - a_k) z_k \lambda_k^{-1/2}$ for a $z \in X$. As in the lectures $R_\gamma f = h$ with $h = \sum_k z_k \lambda_k^{1/2} e_k$. Observe that $h = Q^{1/2} z \in (\ker Q)^\perp$. No further modifications are required. $\square$