Lecture 4

Examples

In this Lecture we present two basic examples that provide in some sense the extreme cases. The first one is $\mathbb{R}^\infty$, which is not even a Banach space, but, being a countable product of real lines, admits a canonical product Gaussian measure which generalises Proposition 1.2.4. The second example is a Hilbert space, whose richer structure allows a more detailed, though simplified, description of the framework. After presenting the relevant Gaussian measures in these spaces, we describe the Reproducing Kernel X^*_γ and the Cameron–Martin space H.

4.1 The product $\mathbb{R}^\infty$

The space $\mathbb{R}^I$ of all the real functions defined in the set I is the only example of non Banach space that is relevant in our lectures. Its importance comes both from some "universal" properties it enjoys and (even more) from the fact that it appears naturally when dealing with stochastic processes, in particular with the Brownian motion. Here, we restrict our attention to the countable case, i.e., $\mathbb{R}^\infty := \mathbb{R}^\mathbb{N}$, the space of all the real sequences. It is obviously a vector space, that we endow with a metrisable locally convex topology coming from the family of semi-norms $p_k(x) := |x_k|$, where $x = (x_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty$. A distance is defined as follows,

$$d(x,y) := \sum_{k=1}^\infty \frac{1}{2^k}\frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

It is easily seen that under this metric every Cauchy sequence is convergent, hence $\mathbb{R}^\infty$ turns out to be a *Fréchet space*. Moreover, the subspace $\mathbb{R}^\infty_c$ of finite sequences, namely the sequences (x_k) that vanish eventually, is isomorphic to the topological dual of $\mathbb{R}^\infty$, under the obvious isomorphism (see exercise 4.1(i))

$$\mathbb{R}^\infty_c \ni (\xi_k) \mapsto f(x) = \sum_{k=1}^\infty \xi_k x_k \text{ (finite sum) }.$$

The elements of $\mathbb{R}^\infty_c$ with rational entries are a countable dense set, hence $\mathbb{R}^\infty$ is separable. The *cylindrical σ-algebra* $\mathscr{E}(\mathbb{R}^\infty)$ is generated by the sets of the form

$$\Big\{x \in \mathbb{R}^\infty : \ (x_1, \dots, x_k) \in B, B \in \mathscr{B}(\mathbb{R}^k)\Big\}.$$

According to Definition 1.1.8, we consider the cylindrical σ–algebra $\mathscr{E}(\mathbb{R}^\infty, F)$ with $F := \{\delta_j, \ j \in \mathbb{N}\}$, generated by the evaluations $\delta_j(x) := x_j$. By Theorem 2.1.1, it coincides with the Borel σ-algebra. According to Remark 1.1.16, we endow $\mathbb{R}^\infty$ with the product measure

$$\gamma := \bigotimes_{k\in\mathbb{N}} \gamma_1 \tag{4.1.1}$$

where γ_1 is the standard Gaussian measure. As in the Banach space case, we say that a probability measure μ in $\mathbb{R}^\infty$ is Gaussian if for every $\xi \in \mathbb{R}^\infty_c$ the measure $\mu \circ \xi^{-1}$ is Gaussian on $\mathbb{R}$. Moreover, it is easily seen that Theorem 2.2.4 holds in $\mathbb{R}^\infty$ as well, see [B, Theorem 2.2.4]. Finally, γ is obviously a Gaussian measure.

Theorem 4.1.1. *The countable product measure γ on $\mathbb{R}^\infty$ is a centred Gaussian measure. Its characteristic function is*

$$\hat\gamma(\xi) = \exp\Big\{-\frac{1}{2}\sum_{k=1}^\infty |\xi_k|^2\Big\} = \exp\Big\{-\frac{1}{2}\|\xi\|^2_{\ell^2}\Big\}, \qquad \xi \in \mathbb{R}^\infty_c, \tag{4.1.2}$$

the Reproducing Kernel is

$$X^*_\gamma = \Big\{f \in L^2(\mathbb{R}^\infty, \gamma) : \ f(x) = \sum_{k=1}^\infty \xi_k x_k, \ (\xi_k) \in \ell^2\Big\}$$

and the Cameron-Martin space H is ℓ^2.

Proof. We compute the characteristic function of γ. For $f(x) = \sum_k \xi_k x_k$, with $x \in \mathbb{R}^\infty$ and $\xi \in \mathbb{R}^\infty_c$ we have

$$\begin{aligned} \hat\gamma(f) &= \int_{\mathbb{R}^\infty} \exp\{if(x)\}\,\gamma(dx) = \int_{\mathbb{R}^\infty} \exp\Big\{i\sum_{k=1}^\infty \xi_k x_k\Big\} \bigotimes_{k=1}^\infty \gamma_1(dx) \\ &= \prod_{k=1}^\infty \int_{\mathbb{R}} \exp\{ix_k\xi_k\}\,\gamma_1(dx_k) = \prod_{k=1}^\infty \exp\Big\{-\frac{1}{2}|\xi_k|^2\Big\} = \exp\Big\{-\frac{1}{2}\|\xi\|^2_{\ell^2}\Big\}. \end{aligned} \tag{4.1.3}$$

According to Theorem 2.2.4, γ is a Gaussian measure with mean $a_\gamma = 0$ and covariance $B_\gamma(\xi, \xi) = \|\xi\|^2_{\ell^2}$.

Let us come to the Cameron-Martin space. Since the mean of γ is 0, for every $f = (\xi_k) \in \mathbb{R}^\infty_c$ we have

$$\begin{aligned}\|f\|^2_{L^2(X,\gamma)} &= \int_X \Big|\sum_k \xi_k x_k\Big|^2 \gamma(dx) = \int_X \left(\sum_k \xi_k^2 x_k^2 + \sum_{j\neq k} \xi_j \xi_k x_j x_k\right) \gamma(dx) \\ &= \sum_k \xi_k^2 \int_{\mathbb{R}} x_k^2 \, \gamma_1(dx_k) + \sum_{j\neq k} \xi_j \int_{\mathbb{R}} x_j \, \gamma(dx_j) \xi_k x_k \int_{\mathbb{R}} x_k \, \gamma(dx_k) \\ &= \sum_k \xi_k^2 = \|\xi\|^2_{\ell^2}\end{aligned}$$

(we recall that all sums are finite) and this shows that X^*_γ, being the closure of $j(X^*) = X^*$, consists of all the functions $f(x) = \sum_k \xi_k x_k$, with $(\xi_k) \in \ell^2$, see Exercise 4.1(ii). On the other hand, for any sequence $h = (h_k) \in \mathbb{R}^\infty$ we have

$$\begin{aligned}|h|_H &= \sup\Big\{ f(h) : \ f \in X^*, \ \|j(f)\|_{L^2(X,\gamma)} \le 1 \Big\} \\ &= \sup\Big\{ \sum_k \xi_k h_k : \ \xi \in \mathbb{R}^\infty_c, \ \sum_k |\xi_k|^2 \le 1 \Big\} = \|h\|_{\ell^2},\end{aligned}$$

and then H coincides with ℓ^2. □

Remark 4.1.2. It is possible to consider more general Gaussian measures on $\mathbb{R}^\infty$, i.e.,

$$\mu = \bigotimes_{k=1}^{\infty} \mathscr{N}(a_k, \lambda_k). \tag{4.1.4}$$

In this case,

$$\hat{\mu}(\xi) = \exp\Big\{ \sum_{k=1}^{\infty} \xi_k a_k - \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \xi_k^2 \Big\}, \qquad \xi = (\xi_k) \in \mathbb{R}^\infty_c, \tag{4.1.5}$$

where as before all the sums contain a finite number of nonzero terms. Let us show that if $(a_k) \in \ell^2$ and $\sum_k \lambda_k < \infty$ then μ is concentrated on ℓ^2, i.e., $\mu(\ell^2) = 1$. Indeed

$$\int_{\mathbb{R}^\infty} \sum_{k=1}^{\infty} |x_k|^2 \, \mu(dx) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} |x_k|^2 \, \mathscr{N}(a_k, \lambda_k)(dx_k) = \sum_{k=1}^{\infty} \lambda_k + \sum_{k=1}^{\infty} |a_k|^2 < \infty.$$

Then $\|x\|_{\ell^2} < +\infty$ μ-a.e in $\mathbb{R}^\infty$, hence $\mu(\ell^2) = 1$.

4.2 The Hilbert space case

Let X be an infinite dimensional separable Hilbert space, with norm $\|\cdot\|_X$ and inner product $\langle \cdot, \cdot \rangle_X$. As usual, we identify X^* with X via the Riesz representation.

We recall that if L is a compact operator on X, the spectrum of L is at most countable and if the spectrum is infinite it consists of a sequence of eigenvalues (λ_k) that can cluster only at 0. If L is compact and self-adjoint, there is an orthonormal basis of X consisting of eigenvectors, see e.g. [Br, Theorem 6.11]. Moreover, L has the representation

$$Lx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle_X, \qquad x \in X, \tag{4.2.1}$$

where $\{e_k : k \in \mathbb{N}\}$ is an orthonormal basis of eigenvectors and $Le_k = \lambda_k e_k$ for any $k \in \mathbb{N}$. If in addition L is nonnegative (namely, $\langle Lx, x \rangle_X \geq 0$ for every $x \in X$) then its eigenvalues are nonnegative and we may define the *square root of* L by

$$L^{1/2}x = \sum_{k=1}^{\infty} \lambda_k^{1/2} \langle x, e_k \rangle_X e_k.$$

The operator $L^{1/2}$ is obviously self-adjoint, and it is also compact. In fact, every operator that can be written in the form

$$Lx = \sum_{k=1}^{\infty} \alpha_k \langle x, e_k \rangle_X e_k,$$

for some orthonormal basis $\{e_k : k \in \mathbb{N}\}$ with $(\alpha_k) \subset \mathbb{R}$, $\lim_{k\to\infty} \alpha_k \to 0$, is compact. Indeed, L is the limit in the operator norm of the sequence of finite rank operators

$$L_n x = \sum_{k=1}^{n} \alpha_k \langle x, e_k \rangle_X e_k,$$

as

$$\|Lx - L_n x\|_X = \Big\| \sum_{k=n+1}^{\infty} \alpha_k \langle x, e_k \rangle_X e_k \Big\|_X \leq \sup_{k>n} |\alpha_k| \|x\|_X,$$

whence $\|L - L_n\|_{\mathcal{L}(X)} \leq \sup_{k>n} |\alpha_k| \to 0$ because $\alpha_k \to 0$ as $k \to \infty$.

A particular class of nonnegative self-adjoint operators which will play an important role is that of *trace-class* operators.

Definition 4.2.1 (Trace-class operators)**.** *A nonnegative self-adjoint operator $L \in \mathcal{L}(X)$ is of* trace-class *or* nuclear *if there is an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of X such that*

$$\sum_{k=1}^{\infty} \langle Le_k, e_k \rangle_X < \infty.$$

It is not hard to see that the sum of the series in definition 4.2.1 is independent of the basis. Indeed, if $\{f_n : n \in \mathbb{N}\}$ is another orthonormal basis, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\langle Le_k, e_k\rangle_X &= \sum_{k=1}^{\infty}\Big\langle L\Big(\sum_{n=1}^{\infty}\langle e_k, f_n\rangle_X f_n\Big), \sum_{m=1}^{\infty}\langle e_k, f_m\rangle_X f_m\Big\rangle_X \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big\langle \langle e_k, f_n\rangle_X Lf_n, \langle e_k, f_m\rangle_X f_m\Big\rangle_X \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big(\sum_{k=1}^{\infty}\langle e_k, f_n\rangle_X \langle e_k, f_m\rangle_X\Big)\langle Lf_n, f_m\rangle_X \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\langle f_n, f_m\rangle_X \langle Lf_n, f_m\rangle_X = \sum_{n=1}^{\infty}\langle Lf_n, f_n\rangle_X,\end{aligned}$$

where we may exchange the order of summation because L is nonnegative.

So, we define tha trace of L as

$$\mathrm{tr}(L) := \sum_{k=1}^{\infty}\langle Le_k, e_k\rangle_X \tag{4.2.2}$$

for any orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of X.

Proposition 4.2.2. *If L is a nonnegative self-adjoint trace-class operator then it is compact.*

Proof. Without loss of generality, assume that $\|L\|_{\mathcal{L}(X)} = 1$. For every $n \in \mathbb{N}$, let X_n be the linear span of $\{e_1, \dots, e_n\}$. Let P_n be the orthogonal projection on X_n and define the finite rank operator $L_n = P_n L P_n$. It is nonnegative and self-adjoint, $\|L_n\|_{\mathcal{L}(X)} \le 1$, and the equality

$$\langle L_n e_k, e_k\rangle_X = \langle Le_k, e_k\rangle_X$$

holds for every $k = 1, \dots, n$. Denoting by $L_n^{1/2}$ the square root of L_n, we have $\|L_n^{1/2}e_k\|_X^2 = \langle Le_k, e_k\rangle_X$ for $k = 1, \dots, n$, whence

$$\sum_{k=1}^{n}\|L_n^{1/2}e_k\|_X^2 \le \mathrm{tr}(L) \quad \forall n \in \mathbb{N}.$$

From $\|L_n^{1/2}\|_{\mathcal{L}(X)} \le 1$ it follows $\|L_n e_k\|_X^2 \le \|L_n^{1/2}\|_X^2\|L_n^{1/2}e_k\|_X^2 \le \|L_n^{1/2}e_k\|_X^2$ for $k = 1 \dots, n$ and

$$\sum_{k=1}^{n}\|L_n e_k\|_X^2 \le \sum_{k=1}^{n}\|L_n^{1/2}e_k\|_X^2 \le \mathrm{tr}(L) \quad \forall n \in \mathbb{N}.$$

Notice that for $k = 1, \dots m$ and $n \ge m$ we have $L_n e_k = P_n L e_k$ and $\lim_{n\to\infty} L_n e_k = Le_k$. Therefore, for every $m \in \mathbb{N}$ we have

$$\sum_{k=1}^{m}\|Le_k\|_X^2 = \lim_{n\to\infty}\sum_{k=1}^{m}\|L_n e_k\|_X^2 \le \mathrm{tr}(L) \quad \forall m \in \mathbb{N}$$

and then letting $m \to \infty$

$$\sum_{k=1}^{\infty} \|Le_k\|_X^2 \le \mathrm{tr}(L). \tag{4.2.3}$$

Using (4.2.3) we prove that L is compact. Let

$$x_n = \sum_{k=1}^{\infty} \langle x_n, e_k\rangle_X e_k \quad \longrightarrow \quad 0 \qquad \text{weakly.}$$

Then, (x_n) is bounded, say $\|x_n\|_X \le M$ for any $n \in \mathbb{N}$. Moreover,

$$Lx_n = \sum_{k=1}^{\infty} \langle x_n, e_k\rangle_X Le_k$$

whence for every $N \in \mathbb{N}$ we have

$$\begin{aligned}\|Lx_n\|_X &\le \sum_{k=1}^{\infty} |\langle x_n, e_k\rangle_X| \|Le_k\|_X = \sum_{k=1}^{N} |\langle x_n, e_k\rangle_X| \|Le_k\|_X + \sum_{k=N+1}^{\infty} |\langle x_n, e_k\rangle_X| \|Le_k\|_X \\ &\le \sum_{k=1}^{N} |\langle x_n, e_k\rangle_X| + M\Big(\sum_{k=N+1}^{\infty} \|Le_k\|_X^2\Big)^{1/2}\end{aligned}$$

and for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that the second term is $< \varepsilon$ because the series in (4.2.3) is convergent. Once this N has been fixed, there is $\nu > 0$ such that the first term is $< \varepsilon$ for $n > \nu$ by the weak convergence of the sequence (x_n) to 0. □

For a complete treatment of the present matter we refer e.g. to [DS1, §VI.5], [DS2, §§XI.6, XI.9].

Let γ be a Gaussian measure in X. According to Theorem 2.2.4 and (2.2.1), (2.2.2) we have

$$\hat{\gamma}(f) = \exp\Big\{ia_\gamma(f) - \frac{1}{2}B_\gamma(f,f)\Big\}, \quad f \in X^*,$$

where the linear mapping $a_\gamma : X^* \to \mathbb{R}$ and the bilinear symmetric mapping $B_\gamma : X^* \times X^* \to \mathbb{R}$ are continuous by Proposition 2.3.3. Then, there are $a \in X$ and a self-adjoint $Q \in \mathcal{L}(X)$ such that $a_\gamma(f) = \langle f, a\rangle_X$ and $B_\gamma(f,g) = \langle Qf, g\rangle_X$ for every f, $g \in X^* = X$ (see Exercise). So,

$$\langle Qf, g\rangle_X = \int_X \langle f, x-a\rangle_X \langle g, x-a\rangle_X \gamma(dx), \quad f,\ g \in X, \tag{4.2.4}$$

and

$$\hat{\gamma}(f) = \exp\Big\{i\langle f, a\rangle_X - \frac{1}{2}\langle Qf, f\rangle_X\Big\}, \quad f \in X. \tag{4.2.5}$$

We denote by $\mathscr{N}(a, Q)$ the Gaussian measure γ whose Fourier transform is given by (4.2.5). As in finite dimension, a is called the mean and Q is called the covariance of γ.

The following theorem is analogous to Theorem 2.2.4, but there is an important difference. In Theorem 2.2.4 a measure is given and we give a criterion to see if it is Gaussian. Instead, in Theorem 4.2.3, we characterise all Gaussian measures in X.

Theorem 4.2.3. *If γ is a Gaussian measure on X then its characteristic function is given by* (4.2.5)*, where $a \in X$ and Q a self-adjoint nonnegative trace-class operator. Conversely, for every $a \in X$ and for every nonnegative self-adjoint trace-class operator Q, the function $\hat{\gamma}$ in* (4.2.5) *is the characteristic function of a Gaussian measure with mean a and covariance operator Q.*

Proof. Let γ be a Gaussian measure and let $\hat{\gamma}$ be its characteristic function, given by (4.2.5). The vector a is the mean of γ by definition, and the symmetry of Q follows from the fact that the bilinear form B_γ is symmetric. By (4.2.4), Q is nonnegative, for every orthonormal basis $\{e_k : k \in \mathbb{N}\}$ we have

$$\sum_{k=1}^{\infty} \langle Qe_k, e_k\rangle_X = \sum_{k=1}^{\infty} \int_X \langle x - a, e_k\rangle_X^2 \, \gamma(dx) = \int_X \|x - a\|_X^2 \, \gamma(dx)$$

which is finite by Corollary 2.3.2. Therefore, Q is a trace-class operator.

Conversely, let Q be a self-adjoint nonnegative trace-class operator. Then Q is given by

$$Qx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k\rangle_X e_k,$$

where $\{e_k : \ k \in \mathbb{N}\}$ is an orthonormal basis of eigenvectors and $Qe_k = \lambda_k e_k$ for any $k \in \mathbb{N}$. Let us consider the measure μ on $\mathbb{R}^\infty$ defined by (4.1.4) and its characteristic function,

$$\hat{\mu}(\xi) = \exp\Big\{ i \sum_{k=1}^{\infty} \xi_k a_k - \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k |\xi_k|^2 \Big\}, \qquad \xi \in \mathbb{R}^\infty_c$$

(recall that the series contains only a finite number of nonzero elements). Let $u : \ell^2 \to X$ be defined by $u(y) = \sum_{k=1}^{\infty} y_k e_k$ (and extended arbitrarily in the μ-negligible set $\mathbb{R}^\infty \setminus \ell^2$, see Remark 4.1.2). Let us show that $\gamma := \mu \circ u^{-1}$ and let us prove that $\gamma = \mathscr{N}(a, Q)$ by computing its characteristic function. For $x \in \mathbb{R}^\infty_c$, setting $z = u(y)$ we have

$$\begin{aligned}
(\widehat{\mu \circ u^{-1}})(x) &= \int_X \exp\{ i \langle z, \sum_{k=1}^{\infty} x_k e_k\rangle_X \} \, (\mu \circ u^{-1})(dz) \\
&= \int_{\mathbb{R}^\infty} \exp\Big\{ i \sum_{k=1}^{\infty} y_k x_k \Big\} \, \mu(dy) \\
&= \int_{\ell^2} \exp\Big\{ i \sum_{k=1}^{\infty} x_k y_k \Big\} \bigotimes_{k=1}^{\infty} \mathscr{N}(a_k, \lambda_k)(dy) \\
&= \exp\Big\{ i \sum_{k=1}^{\infty} x_k a_k - \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k x_k^2 \Big\} \\
&= \exp\Big\{ i \langle x, a\rangle_X - \frac{1}{2} \langle Qx, x\rangle_X \Big\}.
\end{aligned}$$

By Theorem 2.2.4, $(\widehat{\mu \circ u^{-1}})$ is the characteristic function of a unique Gaussian measure with mean a and covariance Q. □

Remark 4.2.4. Since in infinite dimensions the identity is not a trace-class operator, then the function $x \mapsto \exp\{-\frac{1}{2}\|x\|_X^2\}$ cannot be the characteristic function of any Gaussian measure on X.

As a consequence of Theorem 4.2.3 we compute the best constant in Fernique Theorem.

Proposition 4.2.5. *Let $\gamma = \mathscr{N}(a, Q)$ be a Gaussian measure on X and let (λ_k) be the sequence of the eigenvalues of Q. If γ is not a Dirac measure, the integral*

$$\int_X \exp\{\alpha\|x\|_X^2\}\,\gamma(dx)$$

is finite if and only if

$$\alpha < \inf\left\{\frac{1}{2\lambda_k}\ :\ \lambda_k > 0\right\}. \tag{4.2.6}$$

Proof. Let $\{e_k :\ k \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of Q, and $Qe_k = \lambda_k e_k$ for any $k \in \mathbb{N}$. For every $\alpha > 0$, we compute

$$\begin{aligned}
\int_X \exp\{\alpha\|x\|_X^2\}\,\gamma(dx) &= \int_{\mathbb{R}^\infty} \exp\Big\{\alpha\sum_{k=1}^{\infty} x_k^2\Big\} \bigotimes_{k=1}^{\infty} \mathscr{N}(a_k, \lambda_k)(dx) \\
&= \prod_{k=1}^{\infty} \int_{\mathbb{R}} \exp\{\alpha x_k^2\}\,\mathscr{N}(a_k, \lambda_k)(dx_k) \\
&= \prod_{k:\lambda_k=0} \exp\{\alpha a_k^2\} \prod_{k:\lambda_k>0} \frac{1}{\sqrt{2\pi\lambda_k}} \int_{\mathbb{R}} \exp\{\alpha x_k^2\} \exp\Big\{-\frac{1}{2\lambda_k}(x_k - a_k)^2\Big\}\,dx_k.
\end{aligned}$$

If $\alpha \geq (2\lambda_k)^{-1}$ for some $k \in \mathbb{N}$ then the integral with respect to dx_k is infinite, and the function $x \mapsto \exp\{\alpha\|x\|^2\}$ is not in $L^1(X, \gamma)$. If $\alpha < \inf\left\{\frac{1}{2\lambda_k}\ :\ \lambda_k > 0\right\}$ then each integral is finite and we have

$$\int_{\mathbb{R}} \exp\{\alpha x_k^2\} \exp\Big\{-\frac{1}{2\lambda_k}(x_k - a_k)^2\Big\}\,dx_k = \exp\Big\{\frac{\alpha a_k^2}{1 - 2\alpha\lambda_k}\Big\} \frac{1}{\sqrt{1 - 2\alpha\lambda_k}} \tag{4.2.7}$$

for every $k \in \mathbb{N}$. Therefore

$$\begin{aligned}
&\int_X \exp\{\alpha\|x\|_X^2\}\,\gamma(dx) \\
&= \exp\Big\{\alpha\sum_{k:\lambda_k=0} a_k^2\Big\} \exp\Big\{\alpha\sum_{k:\lambda_k>0} \frac{a_k^2}{1 - 2\alpha\lambda_k}\Big\} \exp\Big\{\sum_{k:\lambda_k>0} \log\Big(\frac{1}{\sqrt{1 - 2\alpha\lambda_k}}\Big)\Big\}.
\end{aligned}$$

The convergence of the series follows from $\sum_k \lambda_k < \infty$. □

Let us characterise X_γ^* and the Cameron-Martin space H. By definition, X_γ^* is the closure of $j(X^*)$ in $L^2(X, \gamma)$. Henceforth, if $\gamma = \mathscr{N}(a, Q)$ we fix an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of eigenvectors of Q such that $Qe_k = \lambda_k e_k$ for any $k \in \mathbb{N}$ and for every $x \in X$, $k \in \mathbb{N}$, we set $x_k := \langle x, e_k\rangle_X$.

Theorem 4.2.6. *Let $\gamma = \mathscr{N}(a,Q)$ be a nondegenerate Gaussian measure in X. The space X^*_γ is*

$$X^*_\gamma = \Big\{ f : X \to \mathbb{R} : \ f(x) = \sum_{k=1}^{\infty} (x_k - a_k) z_k \lambda_k^{-1/2}, \ z \in X \Big\} \tag{4.2.8}$$

and the Cameron-Martin space is the range of $Q^{1/2}$, i.e.,

$$H = \Big\{ x \in X : \ \sum_{k=1}^{\infty} x_k^2 \lambda_k^{-1} < \infty \Big\}. \tag{4.2.9}$$

For $h = Q^{1/2} z \in H$, we have

$$\hat{h}(x) = \sum_{k=1}^{\infty} (x_k - a_k) z_k \lambda_k^{-1/2}. \tag{4.2.10}$$

Proof. Let $z \in X$. The series

$$f_n(x) := \sum_{k=1}^{n} x_k z_k \lambda_k^{-1/2}$$

converges in $L^2(X,\gamma)$, since for $m > n$,

$$\|f_m - f_n\|^2_{L^2(X,\gamma)} = \sum_{k=n+1}^{m} \lambda_k^{-1} z_k^2 \int_{\mathbb{R}} (x_k - a_k)^2 \mathscr{N}(a_k, \lambda_k)(dx_k) = \sum_{k=n+1}^{m} z_k^2,$$

and the limit function $f(x) = \sum_{k=1}^{\infty} (x_k - a_k) z_k \lambda_k^{-1/2}$ satisfies

$$\|f\|^2_{L^2(X,\gamma)} = \sum_{k=1}^{\infty} \lambda_k^{-1} z_k^2 \int_{\mathbb{R}} (x_k - a_k)^2 \mathscr{N}(a_k, \lambda_k)(dx_k) = \|z\|^2_X. \tag{4.2.11}$$

Moreover, for every $n \in \mathbb{N}$, $f_n = j(g_n)$, with $g_n \in X^*$,

$$g_n(x) = \sum_{k=1}^{n} \lambda_k^{-1/2} z_k x_k.$$

So, denoting by V the set in the right hand side of (4.2.8), V is contained in the closure of $j(X^*)$ in $L^2(X,\gamma)$, which is precisely X^*_γ.

Let us show that $X^*_\gamma \subset V$. Let $f \in X^*_\gamma$ and let $(w^{(n)}) \subset X$ be a sequence such that $f_n(x) = \langle x - a, w^{(n)} \rangle_X$ converges to f in $L^2(X,\gamma)$. Setting $z^{(n)} := Q^{1/2} w^{(n)}$, we have $f_n(x) = \sum_{k=1}^{\infty} (x_k - a_k) z_k^{(n)} \lambda_k^{-1/2}$, and by (4.2.11),

$$\|z^{(n)} - z^{(m)}\|_X = \|f_n - f_m\|_{L^2(X,\gamma)}, \quad n, \ m \in \mathbb{N},$$

so that $(z^{(n)})$ is a Cauchy sequence, and it converges to some $z \in X$. Then, still by (4.2.11),

$$\begin{aligned}
&\int_X \Big(f(x) - \sum_{k=1}^{\infty}(x_k - a_k) z_k \lambda_k^{-1/2}\Big)^2 \gamma(dx) \\
&= \lim_{n\to\infty} \int_X \Big(\sum_{k=1}^{\infty}(x_k - a_k)(z_k^{(n)} - z_k)\lambda_k^{-1/2}\Big)^2 \gamma(dx) \\
&= \lim_{n\to\infty} \|z^{(n)} - z\|_X^2 = 0,
\end{aligned}$$

so that $f \in V$.

Let us come to the Cameron-Martin space. We know that H is the range of $R_\gamma : X_\gamma^* \to X$, and that $R_\gamma f = h$ iff $\langle f, j(g)\rangle_{L^2(X,\gamma)} = g(h)$ for every $g \in X^*$.

Given any $f \in X_\gamma^*$, $f(x) = \sum_{k=1}^{\infty}(x_k - a_k) z_k \lambda_k^{-1/2}$ for some $z \in X$, and $g \in X^*$, $g(x) = \sum_{k=1}^{\infty} g_k x_k$, we have

$$\int_X f(x) j(g)(x)\gamma(dx) = \int_X \sum_{k=1}^{\infty}(x_k - a_k)^2 z_k \lambda_k^{-1/2} g_k\, \gamma(dx) = \sum_{k=1}^{\infty} z_k \lambda_k^{1/2} g_k.$$

This is equal to $g(h)$ for $h = \sum_{k=1}^{\infty} z_k \lambda_k^{1/2} e_k$, namely $h = Q^{1/2}z$. So, by definition $R_\gamma f = Q^{1/2}z$, hence $H = Q^{1/2}(X)$ and for $h = Q^{1/2}z$ we have $\hat{h}(x) = \sum_{k=1}^{\infty}(x_k - a_k)\lambda_k^{-1/2} z_k$. □

4.3 Exercises

Exercise 4.1. (i) Prove that the dual space of $\mathbb{R}^\infty$ is $\mathbb{R}_c^\infty$.

(ii) Prove that the closure of $\mathbb{R}_c^\infty$ in $\mathbb{R}^\infty$ with respect to the topology induced by the norm $\|\cdot\|_{\ell^2}$ is ℓ^2.

Exercise 4.2. Let X be a real Hilbert space, and let $B : X \times X \to \mathbb{R}$ be bilinear, symmetric and continuous. Prove that there exists a unique self-adjoint operator $Q \in \mathcal{L}(X)$ such that $B(x, y) = \langle Qx, y\rangle_X$, for every x, $y \in X$.

Exercise 4.3. Let $L : \ell^2 \to \ell^2$ be the operator defined by $Lx = (x_2, x_1, x_4, x_3 ....)$, for $x = (x_1, ...x_n, ...) \in \ell^2$. Show that L is self-adjoint and $\langle Le_k, e_k\rangle_{\ell^2} = 0$ and that L is not compact.

Exercise 4.4. Check the computation of the integral in (4.2.7).

Exercise 4.5. Prove that if X is a separable Hilbert space, then $h \in R_\gamma(j(X^*))$ if and only if $h = Qx$, $x \in X$, and that in this case $|h|_H = \|Q^{1/2}x\|_X$.

Exercise 4.6. Modify the proof of Theorem 4.2.6 in order to consider also degenerate Gaussian measures.

Bibliography

[B] V. I. Bogachev: *Gaussian Measures.* American Mathematical Society, 1998.

[Br] H. Brezis: *Functional Analysis, Sobolev spaces and partial differential equations,* Springer, 2011.

[DS1] N. Dunford, J. T. Schwartz: *Linear operators I*, Wiley, 1958.

[DS2] N. Dunford, J. T. Schwartz: *Linear operators II*, Wiley, 1963.