Lecture 3

The Cameron–Martin space

In this Lecture we present the Cameron-Martin space. It consists of the elements $h \in X$ such that the measure $\gamma_h(B) := \gamma(B - h)$ is absolutely continuous with respect to γ. As we shall see, the Cameron-Martin space is fundamental when dealing with the differential structure in X mainly in connection with integration by parts formulae.

3.1 The Cameron–Martin space

We start with the definition of the Cameron–Martin space.

Definition 3.1.1 (Cameron-Martin space). *For every $h \in X$ set*

$$|h|_H := \sup\Big\{f(h) : \ f \in X^*, \ \|j(f)\|_{L^2(X,\gamma)} \le 1\Big\}, \tag{3.1.1}$$

where $j : X^* \to L^2(X,\gamma)$ is the embedding defined in (2.3.4). *The* Cameron-Martin space *is defined by*

$$H := \Big\{h \in X : \ |h|_H < \infty\Big\}. \tag{3.1.2}$$

Calling c the norm of $j : X^* \to L^2(X,\gamma)$, we have

$$\|h\|_X = \sup\{f(h) : \ \|f\|_{X^*} \le 1\} \le \sup\{f(h) : \ \|j(f)\|_{L^2(X,\gamma)} \le c\} = c|h|_H, \tag{3.1.3}$$

and then H is continously embedded in X. We shall see that this embedding is even compact and that the norms $\|\cdot\|_X$ and $|\cdot|_H$ are not equivalent in H.

The Cameron-Martin space inherits a natural Hilbert space structure from the space X^*_γ through the $L^2(X,\gamma)$ Hilbert structure.

Proposition 3.1.2. *An element $h \in X$ belongs to H if and only if there is $\hat{h} \in X^*_\gamma$ such that $h = R_\gamma \hat{h}$. In this case,*

$$|h|_H = \|\hat{h}\|_{L^2(X,\gamma)}. \tag{3.1.4}$$

Therefore $R_\gamma : X^*_\gamma \to H$ is an isometry and H is a Hilbert space with the inner product

$$[h,k]_H := \langle \hat{h}, \hat{k} \rangle_{L^2(X,\gamma)}$$

whenever $h = R_\gamma \hat{h}$, $k = R_\gamma \hat{k}$.

Proof. If $|h|_H < \infty$, we define the map $L : j(X^*) \to \mathbb{R}$ setting

$$L(j(f)) := f(h), \qquad \forall f \in X^*.$$

Such map is well defined since the estimate

$$|f(h)| \leq \|j(f)\|_{L^2(X,\gamma)} |h|_H \tag{3.1.5}$$

implies that if $j(f_1) = j(f_2)$, then $f_1(h) = f_2(h)$. The map L is also continuous with respect to the L^2 topology again by estimate (3.1.5). Then L can be continuously extended to X^*_γ; by the Riesz representation theorem there is a unique $\hat{h} \in X^*_\gamma$ such that the extension (still denoted by L) is given by

$$L(\phi) = \int_X \phi(x)\hat{h}(x)\,\gamma(dx), \qquad \forall \phi \in X^*_\gamma.$$

In particular, for any $f \in X^*$,

$$f(h) = L(j(f)) = \int_X j(f)(x)\hat{h}(x)\,\gamma(dx) = f(R_\gamma \hat{h}),$$

therefore $R_\gamma \hat{h} = h$ and

$$|h|_H = \sup\Big\{ f(h) : \ f \in X^*, \ \|j(f)\|_{L^2(X,\gamma)} \leq 1 \Big\} = \|\hat{h}\|_{L^2(X,\gamma)}.$$

Conversely, if $h = R_\gamma \hat{h}$, then by (2.3.7) for all $f \in X^*$ we have

$$f(h) = f(R_\gamma \hat{h}) = \int_X j(f)(x)\hat{h}(x)\,\gamma(dx) \leq \|\hat{h}\|_{L^2(X,\gamma)} \|j(f)\|_{L^2(X,\gamma)}, \tag{3.1.6}$$

whence $|h|_H < \infty$. □

The space $L^2(X,\gamma)$ (hence its subspace X^*_γ as well) is separable, because X is separable, see e.g. [Br, Theorem 4.13]. Therefore, H, being isometric to a separable space, is separable.

Remark 3.1.3. The map $R_\gamma : X^*_\gamma \to X$ can be defined directly using the Bochner integral through the formula

$$R_\gamma f := \int_X (x-a) f(x)\,\gamma(dx),$$

where a is the mean of γ. We do not assume the knowledge of Bochner intregal. We shall say something about it in one of the following lectures.

Before going on, let us describe the finite dimensional case $X = \mathbb{R}^d$. If $\gamma = \mathscr{N}(a, Q)$ then for $f \in \mathbb{R}^d$ we have

$$\|j(f)\|^2_{L^2(\mathbb{R}^d,\gamma)} = \int_{\mathbb{R}^d} \langle x - a, f\rangle^2_{\mathbb{R}^d} \mathscr{N}(a, Q)(dx) = \langle Qf, f\rangle_{\mathbb{R}^d}$$

and therefore $|h|_H$ is finite if and only if $h \in Q(\mathbb{R}^d)$ and, as a consequence, $H = Q(\mathbb{R}^d)$ is the range of Q. According to the notation introduced in Proposition 3.1.2, if Q is invertible, $h = R_\gamma \hat{h}$ iff $\hat{h}(x) = \langle Q^{-1}h, x\rangle_{\mathbb{R}^d}$. Moreover, if γ is nondegenerate the measures γ_h defined by $\gamma_h(B) = \gamma(B - h)$ are all equivalent to γ in the sense of Section 1.1 and an elementary computation shows that, writing $\gamma_h = \varrho_h \gamma$, we have

$$\varrho_h(x) := \exp\Big\{\langle Q^{-1}h, x\rangle_{\mathbb{R}^d} - \frac{1}{2}|h|^2\Big\} = \exp\Big\{\hat{h}(x) - \frac{1}{2}|h|^2\Big\}.$$

In the infinite dimensional case the situation is completely different. We start with a preliminary result.

Lemma 3.1.4. *For any $g \in X^*_\gamma$, the measure*

$$\mu_g = \exp\Big\{g - \frac{1}{2}\|g\|^2_{L^2(X,\gamma)}\Big\}\gamma$$

is a Gaussian measure with characteristic function

$$\widehat{\mu}_g(f) = \exp\Big\{if(R_\gamma g) + ia_\gamma(f) - \frac{1}{2}\|j(f)\|^2_{L^2(X,\gamma)}\Big\}. \tag{3.1.7}$$

Proof. First of all, we notice that the image of γ under the measurable function $g : X \to \mathbb{R}$ is still a Gaussian measure given by $\mathscr{N}(0, \|g\|^2_{L^2(X,\gamma)})$ thanks to Proposition 2.3.5. Therefore,

$$\int_X \exp\{|g(x)|\}\,\gamma(dx) = \int_{\mathbb{R}} e^{|t|}\,\mathscr{N}(0, \|g\|^2_{L^2(X,\gamma)})(dt) < +\infty,$$

hence $\exp\{|g|\} \in L^1(X, \gamma)$ and μ_g is a finite measure. In addition, μ_g is a probability measure since

$$\begin{aligned}\mu_g(X) =& \int_X \exp\Big\{g(x) - \frac{1}{2}\|g\|^2_{L^2(X,\gamma)}\Big\}\,\gamma(dx)\\ =& \exp\Big\{-\frac{1}{2}\|g\|^2_{L^2(X,\gamma)}\Big\}\int_{\mathbb{R}} e^t \mathscr{N}(0, \|g\|^2_{L^2(X,\gamma)})(dt) = 1.\end{aligned}$$

In order to prove that (3.1.7) holds, we observe that for every $t \in \mathbb{R}$ we have

$$\begin{aligned}&\exp\Big\{-\frac{1}{2}\|g\|^2_{L^2(X,\gamma)}\Big\}\int_X \exp\{i(f(x) - tg(x))\}\,\gamma(dx)\\ =&\exp\Big\{-\frac{1}{2}\|g\|^2_{L^2(X,\gamma)}\Big\}\widehat{\gamma}(f - tg)\\ =&\exp\Big\{-\frac{1}{2}\|g\|^2_{L^2(X,\gamma)}\Big\}\exp\Big\{ia_\gamma(f - tg) - \frac{1}{2}\|j(f - tg)\|^2_{L^2(X,\gamma)}\Big\}\\ =&\exp\Big\{tf(R_\gamma g) - \frac{1 + t^2}{2}\|g\|^2_{L^2(X,\gamma)} + ia_\gamma(f) - \frac{1}{2}\|j(f)\|^2_{L^2(X,\gamma)}\Big\}.\end{aligned}$$

So, the entire holomorphic functions

$$z \mapsto \exp\Big\{-\frac{1}{2}\|g\|^2_{L^2(X,\gamma)}\Big\} \int_X \exp\{i(f(x) - zg(x))\}\,\gamma(dx)$$
$$z \mapsto \exp\Big\{zf(R_\gamma g) - \frac{1+z^2}{2}\|g\|^2_{L^2(X,\gamma)} + ia_\gamma(f) - \frac{1}{2}\|j(f)\|^2_{L^2(X,\gamma)}\Big\}$$

coincide for $z \in \mathbb{R}$, hence they coincide in $\mathbb{C}$. In particular, taking $z = i$ we obtain

$$\widehat{\mu}_g(f) = \exp\Big\{ia_\gamma(f) - \frac{1}{2}\|j(f)\|^2_{L^2(X,\gamma)} + iR_\gamma g(f)\Big\}.$$

□

Theorem 3.1.5 (Cameron-Martin Theorem)**.** *For $h \in X$, define the measure $\gamma_h(B) := \gamma(B-h)$. If $h \in H$ the measure γ_h is equivalent to γ and $\gamma_h = \varrho_h\gamma$, with*

$$\varrho_h(x) := \exp\Big\{\hat{h}(x) - \frac{1}{2}|h|^2_H\Big\}, \tag{3.1.8}$$

where $\hat{h} = R_\gamma^{-1}h$. If $h \notin H$ then $\gamma_h \perp \gamma$. Hence, $\gamma_h \approx \gamma$ if and only if $h \in H$.

Proof. For $h \in H$, let us compute the characteristic function of γ_h. For any $f \in X^*$ we have

$$\begin{aligned}\hat{\gamma}_h(f) &= \int_X \exp\{if(x)\}\,\gamma_h(dx) = \int_X \exp\{if(x+h)\}\,\gamma(dx)\\ &= \exp\Big\{if(R_\gamma\hat{h}) + ia_\gamma(f) - \frac{1}{2}\|j(f)\|^2_{L^2(X,\gamma)}\Big\}, \qquad f \in X^*.\end{aligned}$$

Taking into account Lemma 3.1.4 and Proposition 2.1.2, we obtain that $\gamma_h = \varrho_h\gamma$, where the density ϱ_h is given by (3.1.8).

Now, let us see that if $h \notin H$ then $\gamma_h \perp \gamma$. To this aim, let us first consider the 1-dimensional case. If γ is a Dirac measure in $\mathbb{R}$, then $\gamma_h \perp \gamma$ for any $h \neq 0$ and $|\gamma - \gamma_h|(\mathbb{R}) = 2$. Otherwise, if $\gamma = \mathscr{N}(a,\sigma^2)$ is a nondegenerate Gaussian measure in $\mathbb{R}$, then $\gamma_h \ll \gamma$ with $\frac{d\gamma_h}{d\gamma}(t) = \exp\{-\frac{h^2}{2\sigma^2} + \frac{h(t-a)}{\sigma^2}\}$. We can apply Hellinger Theorem 1.1.10 with $\lambda = \gamma$, whence by Exercise 3.2

$$H(\gamma,\gamma_h) = \exp\Big\{-\frac{h^2}{8\sigma^2}\Big\}, \tag{3.1.9}$$

and then (1.1.7) implies

$$|\gamma - \gamma_h|(\mathbb{R}) \geq 2\left(1 - \exp\Big\{-\frac{1}{8\sigma^2}h^2\Big\}\right). \tag{3.1.10}$$

In any case, (3.1.10) holds true.

Let us go back to X. For every $f \in X^*$, using just the definition, it is immediate to verify that $\gamma_h \circ f^{-1} = (\gamma \circ f^{-1})_{f(h)}$ and

$$|\gamma \circ f^{-1} - (\gamma \circ f^{-1})_{f(h)}|(\mathbb{R}) \leq |\gamma - \gamma_h|(X);$$

If $h \notin H$, there exists a sequence $(f_n) \subset X^*$ with $\|j(f_n)\|_{L^2(X,\gamma)} = 1$ and $f_n(h) \geq n$. By (3.1.10) we obtain

$$\begin{aligned}|\gamma - \gamma_h|(X) \geq& |(\gamma \circ f_n^{-1}) - (\gamma \circ f_n^{-1})_{f_n(h)}|(\mathbb{R}) \geq 2\left(1 - \exp\Big\{-\frac{1}{8}f_n(h)^2\Big\}\right)\\ \geq& 2\left(1 - \exp\Big\{-\frac{1}{8}n^2\Big\}\right).\end{aligned}$$

This implies that $|\gamma - \gamma_h|(X) = 2$, hence by Corollary 1.1.11, $\gamma_h \perp \gamma$. □

From now on, we denote by $B^H(0,r)$ the open ball of centre 0 and radius r in H and by $\overline{B}^H(0,r)$ its closure in H. In the proof of Theorem 3.1.8 we need the following result.

Proposition 3.1.6. *If $A \in borel(X)$ is such that $\gamma(A) > 0$, then there is $r > 0$ such that $B^H(0,r) \subset A - A$.*

Proof. Let us introduce the function $H \ni h \mapsto \phi(h) := \gamma\big((A+h) \cap A\big)$, i.e.,

$$\phi(h) = \gamma\big((A+h) \cap A\big) = \int_X 1\!\!1_A(x-h)1\!\!1_A(x)\,\gamma(dx).$$

We claim that

$$\liminf_{|h|_H \to 0} \phi(h) > 0.$$

Let us first assume that A is open; then

$$1\!\!1_A(x) = \sup\{\varphi(x) : \varphi \in C(X), 0 \leq \varphi(x) \leq 1\!\!1_A(x)\}.$$

In particular we consider the sequence of functions

$$\varphi_n(x) = \min\Big\{n\mathrm{dist}(x, A^c), 1\Big\}.$$

For every n, $\varphi_n = 0$ on A^c, $\varphi_n = 1$ on $\{x \in A : \mathrm{dist}(x, A^c) \geq \frac{1}{n}\}$, it is Lipschitz continuous and $\varphi_n(x) \to 1\!\!1_A(x)$ for any $x \in X$. By continuity and by the fact that $\|h\|_X \leq c|h|_H$, for any $x \in X$

$$\lim_{|h|_H \to 0} \varphi_n(x-h) = \varphi_n(x).$$

Then by the Fatou Lemma we have

$$\begin{aligned}\int_X \varphi_n(x)^2\gamma(dx) =& \int_X \lim_{|h|_H \to 0} \varphi_n(x-h)\varphi_n(x)\gamma(dx)\\ \leq& \liminf_{|h|_H \to 0} \int_X 1\!\!1_A(x-h)1\!\!1_A(x)\gamma(dx).\end{aligned}$$

Letting $n \to +\infty$, by Lebesgue Dominated Convergence Theorem, we obtain

$$\gamma(A) = \lim_{n\to+\infty} \int_X \varphi_n(x)^2\,\gamma(dx) \leq \liminf_{|h|_H \to 0} \int_X 1\!\!1_A(x-h)1\!\!1_A(x)\gamma(dx),$$

and we have proved the claim for A open.

Let us now consider an arbitrary $A \in \mathscr{B}(X)$; in the proof of Proposition 1.1.5 we have seen that for any $\varepsilon > 0$ there exists an open set $A_\varepsilon \supset A$ with $\gamma(A_\varepsilon \setminus A) < \varepsilon$. Taking into account that $h \in H$, we get

$$\begin{aligned}
\int_X \mathbb{1}_A(x-h)\mathbb{1}_A(x)\,\gamma(dx) = \\
&= \int_X \mathbb{1}_{A_\varepsilon}(x-h)\mathbb{1}_A(x)\,\gamma(dx) + \int_X (\mathbb{1}_A(x-h) - \mathbb{1}_{A_\varepsilon}(x-h))\mathbb{1}_A(x)\,\gamma(dx) \\
&= \int_X \mathbb{1}_{A_\varepsilon}(x-h)\mathbb{1}_{A_\varepsilon}(x)\,\gamma(dx) + \int_X \mathbb{1}_{A_\varepsilon}(x-h)(\mathbb{1}_A(x) - \mathbb{1}_{A_\varepsilon}(x))\,\gamma(dx) + \\
&\quad + \int_X (\mathbb{1}_A(x-h) - \mathbb{1}_{A_\varepsilon}(x-h))\mathbb{1}_A(x)\,\gamma(dx) \\
&\geq \int_X \mathbb{1}_{A_\varepsilon}(x-h)\mathbb{1}_{A_\varepsilon}(x)\gamma(dx) - \int_X |\mathbb{1}_A(x) - \mathbb{1}_{A_\varepsilon}(x)|\,\gamma(dx) + \\
&\quad - \int_X |\mathbb{1}_A(x-h) - \mathbb{1}_{A_\varepsilon}(x-h)|\,\gamma(dx) \\
&= \int_X \mathbb{1}_{A_\varepsilon}(x-h)\mathbb{1}_{A_\varepsilon}(x)\gamma(dx) - \gamma(A_\varepsilon \setminus A) + \\
&\quad - \int_X |\mathbb{1}_A(x) - \mathbb{1}_{A_\varepsilon}(x)| \exp\Big\{-\hat{h}(x) - \frac{1}{2}|h|_H^2\Big\}\,\gamma(dx).
\end{aligned}$$

Now we claim that

$$\lim_{|h|_H \to 0} \int_X |\mathbb{1}_A(x) - \mathbb{1}_{A_\varepsilon}(x)| \exp\Big\{-\hat{h}(x) - \frac{1}{2}|h|_H^2\Big\}\,\gamma(dx) = \gamma(A_\varepsilon \setminus A). \tag{3.1.11}$$

Once (3.1.11) is proved, it implies

$$\begin{aligned}
\liminf_{|h|_H \to 0} \int_X \mathbb{1}_A(x-h)\mathbb{1}_A(x)\gamma(dx) &\geq \liminf_{|h|_H \to 0} \int_X \mathbb{1}_{A_\varepsilon}(x-h)\mathbb{1}_{A_\varepsilon}(x)\gamma(dx) + \\
&\quad - 2\gamma(A_\varepsilon \setminus A) \geq \gamma(A_\varepsilon) - 2\varepsilon \geq \gamma(A) - 2\varepsilon > 0
\end{aligned}$$

if $\varepsilon < \gamma(A)/2$. Then there is $r > 0$ such that $\phi(h) > 0$ for $|h|_H < r$ and therefore for any $|h|_H < r$, $(A + h) \cap A \neq \emptyset$, so that $B^H(0, r) \subset A - A$.

To prove that (3.1.11) holds, we notice that since the image measure of γ under $\hat{h}$ is $\mathscr{N}(0, |h|_H^2)$, then

$$\int_X \Big|\exp\Big\{-\hat{h}(x) - \frac{1}{2}|h|_H^2\Big\} - 1\Big|\,\gamma(dx) = \int_{\mathbb{R}} \Big|\exp\Big\{-t|h|_H - \frac{1}{2}|h|_H^2\Big\} - 1\Big|\,\gamma_1(dt),$$

and the right hand side vanishes as $|h|_H \to 0$ by the Dominated Convergence Theorem. □

We give the following technical result that we shall need for instance in the proof of Theorem 3.1.8; it will be rephrased with a probabilistic language in the sequel.

Lemma 3.1.7. *Let $f, g \in X^*$ and set $T : X \to \mathbb{R}^2$, $T(x) := (f(x), g(x))$. Then*

$$\gamma \circ T^{-1} = (\gamma \circ f^{-1}) \otimes (\gamma \circ g^{-1})$$

iff $j(f)$ and $j(g)$ are orthogonal in $L^2(X, \gamma)$.

Proof. We just compute the characteristic function. For every $\xi \in \mathbb{R}^2$ we have

$$\begin{aligned}\widehat{\gamma \circ T^{-1}}(\xi) &= \int_X \exp\{i\xi(T(x))\}\gamma(dx) = \int_X \exp\{i(\xi_1 f + \xi_2 g)(x)\}\gamma(dx) \\ &= \exp\left\{i\xi_1 a_\gamma(f) + i\xi_2 a_\gamma(g) - \frac{1}{2}\|j(\xi_1 f + \xi_2 g)\|^2_{L^2(X,\gamma)}\right\}.\end{aligned}$$

On the other hand, if $\mu = (\gamma \circ f^{-1}) \otimes (\gamma \circ g^{-1})$, then

$$\begin{aligned}\widehat{\mu}(\xi) &= (\widehat{\gamma \circ f^{-1}})(\xi_1)(\widehat{\gamma \circ g^{-1}})(\xi_2) \\ &= \exp\left\{i\xi_1 a_\gamma(f) + i\xi_2 a_\gamma(g) - \frac{\xi_1^2}{2}\|j(f)\|^2_{L^2(X,\gamma)} - \frac{\xi_2^2}{2}\|j(g)\|^2_{L^2(X,\gamma)}\right\},\end{aligned}$$

whence the conclusion, since

$$\|j(\xi_1 f + \xi_2 g)\|^2_{L^2(X,\gamma)} = \xi_1^2\|j(f)\|^2_{L^2(X,\gamma)} + \xi_2^2\|j(g)\|^2_{L^2(X,\gamma)}$$

if and only if $\langle f, g\rangle_{L^2(X,\gamma)} = 0$. □

In the proof of the following result we deal with a nonmetrisable topology, and this requires the use of *nets*. Let us recall that a *directed set* is an ordered set (we denote by $\geq$ the order relation) in which any couple of elements possess a common majorant. A *net* is a function whose domain is a directed set. If S is a topological space, a net $(x_\alpha)_{\alpha\in\mathbb{A}} \subset S$ is said to converge to $x \in S$ iff for any neighbourhood U of x there is $\alpha_0 \in \mathbb{A}$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. A point x belongs to the closure of a set $E \subset S$ iff there is a net $(x_\alpha)_{\alpha\in\mathbb{A}} \subset A$ converging to x.

Theorem 3.1.8. *Let γ be a Gaussian measure in a separable Banach space X, and let H be its Cameron–Martin space. The following statements hold.*

(i) The unit ball $B^H(0,1)$ of H is relatively compact in X and hence the embedding $H \hookrightarrow X$ is compact.

(ii) H is the intersection of all the Borel full measure subspaces of X.

(iii) If X^*_γ is infinite dimensional then $\gamma(H) = 0$.

Proof. (i) By inequality (3.1.3), the ball $\overline{B}^H(0,1)$ is bounded in X. Let us prove that it is weakly closed. To this aim, consider a net $(h_\alpha) \subset \overline{B}^H(0,1)$ weakly converging to $h \in X$. For every $f \in X^*$ with $\|j(f)\|_{L^2(X,\gamma)} \leq 1$ the inequality $|f(h)| \leq 1$ holds because $f(h) = \lim_\alpha f(h_\alpha)$ and $\|f\|_{X^*} \leq \|j(f)\|_{L^2(X,\gamma)}$. Hence $\overline{B}^H(0,1)$ is weakly closed, then it is

closed in X since it is convex, see e.g. [Br, Theorem 3.7]. To prove that the embedding of H in X is compact, it is sufficient to prove that $\overline{B}^H(0,r)$ is compact in X for some $r > 0$. Fix any compact set $K \subset X$ with $\gamma(K) > 0$; by Lemma 3.1.6 there is $r > 0$ such that the ball $B^H(0,r)$ is contained in the compact set $K - K$, which implies that $\overline{B}^H(0,r)$ is contained in $K - K$ and the proof is complete.
(ii) Let V be a subspace of X with $\gamma(V) = 1$ and fix $h \in H$; by Theorem 3.1.5,

$$\begin{aligned}\gamma(V-h) =&\gamma_h(V) = \int_V \exp\Big\{\hat{h}(x) - \frac{1}{2}|h|_H^2\Big\}\,\gamma(dx)\\ =&\int_X \exp\Big\{\hat{h}(x) - \frac{1}{2}|h|_H^2\Big\}\,\gamma(dx) = 1.\end{aligned}$$

This implies that $h \in V$, since otherwise $V \cap (V - h) = \emptyset$ and we would have

$$1 = \gamma(X) \geq \gamma(V) + \gamma(V-h) = 2,$$

a contradiction. Therefore, $H \subset V$ for all subspaces V of full measure.

To prove that the intersection of all subspaces of X with full measure is contained in H, fixed any $h \notin H$, we construct a full measure subspace V such that $h \notin V$. If $h \notin H$, then $|h|_H = +\infty$ and there is a sequence $(f_n) \subset X^*$ with $\|j(f_n)\|_{L^2(X,\gamma)} = 1$ and $f_n(h) \geq n$. Since

$$\sum_{n=1}^\infty \frac{1}{n^2}\int_X |j(f_n)(x)|\,\gamma(dx) \leq \sum_{n=1}^\infty \frac{1}{n^2}\|j(f_n)\|_{L^2(X,\gamma)} < \infty$$

the space (which is a Borel set, see Exercise 3.1)

$$V := \Big\{x \in X : \ \text{ the series } \sum_{n=1}^\infty \frac{1}{n^2} j(f_n)(x) \ \text{ is convergent } \Big\} \tag{3.1.12}$$

has full measure, and $h \notin V$.
(iii) Let us assume that X^*_γ is infinite dimensional. Then, there exists an orthonormal basis $\{f_n : \ n \in \mathbb{N}\}$ of X^*_γ; in particular for any $n \in \mathbb{N}$, $\gamma \circ f_n^{-1} = \mathscr{N}(0,1)$. For every $M > 0$ and $n \in \mathbb{N}$ we have

$$\gamma(\{x \in X : \ |f_n(x)| \leq M\}) = \mathscr{N}(0,1)(-M, M) =: a_M < 1;$$

as a consequence, since the functions f_n mutually are orthogonal, by Lemma 3.1.7 we have

$$\gamma(\{x \in X : \ |f_k(x)| \leq M \ \text{ for } k = 1,\dots,n\}) = a_M^n \to 0 \quad \text{ as } n \to \infty$$

and then

$$\gamma\Big(\Big\{x \in X : \ \sup_{n\in\mathbb{N}} |f_n(x)| \leq M\Big\}\Big) = \gamma\Big(\bigcap_{n\in\mathbb{N}} \{x \in X : \ |f_k(x)| \leq M, k = 1,\dots,n\}\Big) = 0.$$

Since $\{f_n:\ n\in\mathbb{N}\}$ is a basis of X^*_γ, for any $h\in H$ we have

$$|h|_H^2=\|\hat h\|^2_{L^2(X,\gamma)}=\sum_{n=1}^\infty\langle f_n,\hat h\rangle^2_{L^2(X,\gamma)}=\sum_{n=1}^\infty f_n(h)^2.$$

Therefore

$$H=\Big\{x\in X:\ \sum_{n=1}^\infty f_n(x)^2<\infty\Big\}\subset\bigcup_{M>0}\Big\{x\in X:\ \sup_{n\in\mathbb{N}}|f_n(x)|\le M\Big\}$$

and it has measure 0. □

We close this lecture with a couple of properties of the reproducing kernel and of the Cameron–Martin space. First we show that it is always possible to consider orthonormal basis in X^*_γ made by elements of $j(X^*)$; this fact can be very useful in some proofs. Then we see that the norm of the space X is somehow irrelevant in the theory, in the sense that the Cameron–Martin space remains unchanged if we replace the norm of X by a weaker norm.

Lemma 3.1.9. *There exists an orthonormal basis of X^*_γ contained in $j(X^*)$.*

Proof. Let $\{f_k:\ k\in\mathbb{N}\}$ be an orthonormal basis of X^*_γ. In its turn, every f_k is the $L^2(X,\gamma)$-limit of a sequence of elements $j(g_n^{(k)})$ with $g_n^{(k)}\in X^*$. Let us enumerate the set $\{j(g_n^{(k)}):\ k,\ n\in\mathbb{N}\}$, for instance by the diagonal procedure. On span $\{j(g_n^{(k)}):\ k,\ n\in\mathbb{N}\}$ we construct an orthonormal basis $\mathcal{V}$, by the Gram–Schmidt procedure (see e.g. [L, Theorem V.2.1]). The linear combinations of the elements of such a basis approach every $j(g_n^{(k)})$ and hence every f_k in $L^2(X,\gamma)$. Therefore, the linear space spanned by $\mathcal{V}$ is dense in X^*_γ. □

Proposition 3.1.10. *Let γ be a Gaussian measure on a Banach space X. Let us assume that X is continuously embedded in another Banach space Y, i.e., there exists a continuous injection $i:X\to Y$. Then the Cameron–Martin space H associated with the measure γ is isomorphic to the Cameron–Martin space H_Y associated with the image measure $\gamma_Y:=\gamma\circ i^{-1}$ in Y.*

Proof. Let $f\in Y^*$; then $f\circ i\in X^*$ by the continuity of the injection i. Moreover

$$a_\gamma(f\circ i)=\int_X f(i(x))\,\gamma(dx)=\int_Y f(y)\,\gamma_Y(dy)=a_{\gamma_Y}(f).$$

Denoting by $j_Y:Y^*\to Y^*_{\gamma_Y}$ the embedding of Y^* into $L^2(Y,\gamma_Y)$, we have $j(f\circ i)=j_Y(f)\circ i$ and

$$\|j(f\circ i)\|^2_{L^2(X,\gamma)}=\int_X j(f\circ i)(x)^2\,\gamma(dx)=\int_Y j_Y(f)(y)^2\,\gamma_Y(dy)=\|j_Y(f)\|^2_{L^2(Y,\gamma_Y)}.$$

We prove now that $i:H\to H_Y$ is an isometry. First of all, $i(h)\in H_Y$ for any $h\in H$ since for any $f\in Y^*$

$$|f(i(h))|=|(f\circ i)(h)|\le\|j(f\circ i)\|_{L^2(X,\gamma)}|h|_H$$

and then

$$|i(h)|_{H_Y} = \sup\{f(i(h)) : f \in Y^*,\ \|j_Y(f)\|_{L^2(Y,\gamma_Y)} \le 1\} \le |h|_H < +\infty.$$

Hence $i(H) \subset H_Y$ and

$$|i(h)|_{H_Y} \le |h|_H. \tag{3.1.13}$$

We prove now the inclusion $H_Y \subset i(H)$; since $i(X)$ has full measure in Y, we have $H_Y \subset i(X)$ by statement (ii) of Theorem 3.1.8. Then, for any $h_Y \in H_Y$, there exists a unique $h \in X$ with $i(h) = h_Y$; since

$$\gamma_Y(B - h_Y) = \gamma(i^{-1}(B) - h),$$

then $h_Y \in H_Y$ if and only if $h \in H$. In this case

$$\begin{aligned}\gamma_Y(B - h_Y) &= \int_B \exp\Big\{\hat h_Y(y) - \frac{1}{2}|h_Y|^2_{H_Y}\Big\}\,\gamma_Y(dy)\\ &= \int_{i^{-1}(B)} \exp\Big\{\hat h_Y(i(x)) - \frac{1}{2}|h_Y|^2_{H_Y}\Big\}\,\gamma(dx)\end{aligned}$$

is equal to

$$\gamma(i^{-1}(B) - h) = \int_{i^{-1}(B)} \exp\Big\{\hat h(x) - \frac{1}{2}|h|^2_H\Big\}\,\gamma(dx).$$

This implies

$$\hat h_Y(i(x)) - \frac{1}{2}|h_Y|^2_{H_Y} = \hat h(x) - \frac{1}{2}|h|^2_H \tag{3.1.14}$$

for γ–a.e. $x \in X$. By (3.1.13) we obtain $\hat h_Y(i(x)) - \hat h(x) \le 0$ for γ–a.e. $x \in X$, and then, since

$$\int_X (\hat h_Y(i(x)) - \hat h(x))\,\gamma(dx) = \int_Y \hat h_Y(y)\,\gamma_Y(dy) - \int_X \hat h(x)\,\gamma(dx) = 0,$$

we conclude that $\hat h_Y(i(x)) = \hat h(x)$ for γ–a.e. $x \in X$ and then by (3.1.14) $|h_Y|_{H_Y} = |i(h)|_{H_Y} = |h|_H$. □

3.2 Exercises 3

Exercise 3.1. Prove that the space V in (3.1.12) is a Borel set.

Exercise 3.2. Show that (3.1.9) holds.

Exercise 3.3. Let γ be the measure on $\mathbb{R}^2$ defined by

$$\gamma(B) = \gamma_1(\{x \in \mathbb{R} : (x, 0) \in B\}), \qquad B \in \mathscr{B}(\mathbb{R}^2).$$

Prove that the Cameron–Martin space H is given by $\mathbb{R} \times \{0\}$.

Exercise 3.4. Let γ, μ be equivalent Gaussian measures in X, and denote by H_γ, H_μ the associated Cameron–Martin spaces. Prove that for every $x \in X$, $x \in H_\gamma$ iff $x \in H_\mu$, and if in addition γ and μ are centered, then $X^*_\gamma = X^*_\mu$. Prove that if γ, μ are centred Gaussian measures in X such that $\gamma \perp \mu$, then $\gamma_x \perp \mu_y$, for all x, $y \in X$.

Exercise 3.5. Prove that the Cameron–Martin space is invariant by translation, i.e. for any $x \in X$, the measure

$$\gamma_x(B) = \gamma(B - x), \qquad \forall B \in \mathscr{B}(X)$$

has the same Cameron–Martin space as γ even when $\gamma_x \perp \gamma$.

Bibliography

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