Exercise 3.1

We will prove that

$$\begin{aligned} V &:= \Big\{ x \in X : \text{ the series } \sum_{n=1}^{\infty} \frac{1}{n^2} j(f_n)(x) \text{ is convergent} \Big\} \\ &= \Big\{ x \in X : \text{ the sequence } \Big(\sum_{n=1}^{m} \frac{1}{n^2} j(f_n)(x) \Big)_{m \in \mathbb{N}} \text{ is a Cauchy sequence} \Big\} \end{aligned}$$

is a Borel set. Here, X denotes a separable Banach space, $j : X^* \to L^2(X, \gamma)$ the mapping $j(f) := f - a_\gamma(f)$, and $(f_n)_{n \in \mathbb{N}} \subset X^*$ a sequence with $\|j(f_n)\|_{L^2(X,\gamma)} = 1$ and $f_n(h) \geq n$.

Writing the Cauchy sequence property using quantors reveals that $x \in V$ if and only if

$$\forall j \in \mathbb{N} \; \exists n_0 \in \mathbb{N} \; \forall k \in \mathbb{N} \; \forall m \in \mathbb{N} \text{ with } m \geq k \geq n_j : \Big| \sum_{n=k}^{m} \frac{1}{n^2} j(f_n)(x) \Big| < \frac{1}{j}.$$

Thus, V can be written as

$$V = \bigcap_{j=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{k=n_0}^{\infty} \bigcap_{m=k}^{\infty} \Big\{ x \in X : \Big| \sum_{n=k}^{m} \frac{1}{n^2} j(f_n)(x) \Big| < \frac{1}{j} \Big\}.$$

Consequently, it remains to show that

$$V_{j,k,m} := \Big\{ x \in X : \Big| \sum_{n=k}^{m} \frac{1}{n^2} j(f_n)(x) \Big| < \frac{1}{j} \Big\}$$

is a Borel set. As $f_n \in X^*$ is continuous and $a_\gamma(f_n)$ is simply a number, the function $x \mapsto j(f_n)(x)$ is continuous from X to $\mathbb{R}$ and hence $V_{j,k,m}$ is open.

Exercise 3.2

In this exercise, $\gamma = \mathcal{N}(a, \sigma^2)$ denotes a nondegenerate Gaussian measure on $\mathbb{R}$ and for $h \neq 0$ the measure γ_h denotes the translation of γ given by $\gamma_h(B) := \gamma(B - h)$ for any Borel set B. By virtue of

$$\begin{aligned} \gamma_h(B) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{B-h} \mathrm{e}^{-\frac{(t-a)^2}{2\sigma^2}} \, \mathrm{d}t \\ &\overset{t=x-h}{=} \frac{1}{\sigma\sqrt{2\pi}} \int_{B} \mathrm{e}^{-\frac{(x-a-h)^2}{2\sigma^2}} \, \mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{B} \mathrm{e}^{-\frac{h^2-2h(x-a)}{2\sigma^2}} \cdot \mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}} \, \mathrm{d}x \end{aligned}$$

it follows that γ_h is absolutely continuous with respect to γ with $\frac{\mathrm{d}\gamma_h}{\mathrm{d}\gamma}(x) = \exp\{-\frac{h^2}{2\sigma^2} + \frac{h(x-a)}{\sigma^2}\}$.

Next, we prove that $H(\gamma, \gamma_h) = \exp\{-\frac{h^2}{8\sigma^2}\}$ holds true. Using the definition of $H(\gamma, \gamma_h)$ (see Hellinger's Theorem in Lecture 1), we derive by a direct computation

$$
\begin{aligned}
H(\gamma, \gamma_h) &= \int_{\mathbb{R}} \sqrt{\frac{\mathrm{d}\gamma}{\mathrm{d}\gamma}(x)\frac{\mathrm{d}\gamma_h}{\mathrm{d}\gamma}(x)}\,\gamma(\mathrm{d}x) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{h^2}{4\sigma^2}+\frac{h(x-a)}{2\sigma^2}-\frac{(x-a)^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{h^2}{4\sigma^2}} \int_{\mathbb{R}} \mathrm{e}^{\frac{1}{2\sigma^2}[ht-t^2]}\,\mathrm{d}t,
\end{aligned}
$$

the last equality following by the change of variables $x = t - a$. Adding $h^2/4 - h^2/4$ in order to complete the square inside the exponential function yields

$$
\begin{aligned}
&= \mathrm{e}^{-\frac{h^2}{8\sigma^2}} \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{[t-\frac{h}{2}]^2}{2\sigma^2}}\,\mathrm{d}t \\
&= \mathrm{e}^{-\frac{h^2}{8\sigma^2}},
\end{aligned}
$$

the latter equality following due to the fact that the integrand is the density of a Gaussian measure.

Exercise 3.3

In this exercise, γ is the measure on $\mathbb{R}^2$ given by

$$
\gamma(B) := \gamma_1(\{x \in \mathbb{R} : (x, 0) \in B\}), \qquad B \in \mathfrak{B}(\mathbb{R}^2),
$$

where $\gamma_1 := \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{|x|^2}{2}}\lambda_1$ is the standard Gaussian measure. To compute the Cameron-Martin space we rely on the fact that if $\gamma = \mathcal{N}(a, Q)$, for some $a \in \mathbb{R}^2$ and a non-negative, symmetric matrix Q, the Cameron-Martin space is given as the range of Q, see the paragraph at the top of page 29. In order to calculate Q we compute the characteristic function of γ. By definition of the characteristic function

$$
\widehat{\gamma}(\xi) = \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}x\cdot\xi}\gamma(\mathrm{d}x).
$$

Since $\mathbb{R} \times [\mathbb{R} \setminus \{0\}]$ has measure zero

$$= \int_{\mathbb{R}\times\{0\}} e^{ix_1\xi_1}\gamma(dx)$$

and by definition of γ

$$= \int_{\mathbb{R}} e^{ix_1\xi_1}\gamma_1(dx_1).$$

Finally, Remark 1.2.2 yields

$$= e^{-\frac{\xi_1^2}{2}}.$$

We conclude that Q is given by

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This proves $H = Q(\mathbb{R}^2) = \mathbb{R} \times \{0\}$.

Exercise 3.3 (alternative approach)

We consider the measure γ on $\mathbb{R}^2$ defined by

$$\gamma(B) = \gamma_1(\{x \in \mathbb{R} : (x, 0) \in B\}), \quad B \in \mathcal{B}(\mathbb{R}^2).$$

In oder to show that the Cameron-Martin space H is given by $\mathbb{R} \times \{0\}$, we will use Theorem 3.1.8(ii), which tells us that H is the intersection of all the Borel full measure subspaces of $\mathbb{R}^2$.

At first, we remark that

$$\gamma(\mathbb{R} \times \{0\}) = \gamma_1(\mathbb{R}) = 1$$

because γ_1 is a probability measure on $\mathbb{R}$. Therefore, we have $H \subseteq \mathbb{R} \times \{0\}$ by Theorem 3.1.8(ii).

Now let $B \in \mathcal{B}(\mathbb{R}^2)$ be a full measure subspace of $\mathbb{R}^2$. We want to show that $\mathbb{R} \times \{0\} \subseteq B$. We assume that this is not the case. But then the vector space $(\mathbb{R} \times \{0\}) \cap B$ has to be a proper subspace of $\mathbb{R} \times \{0\}$. Hence $(\mathbb{R} \times \{0\}) \cap B = \{(0, 0)\}$.

Because $\gamma(\{(0, 0)\}) = 0$, we obtain

$$\begin{aligned} 1 = \gamma(\mathbb{R}^2) &\geq \gamma((\mathbb{R} \times \{0\}) \cup B) \\ &= \gamma(\mathbb{R} \times \{0\}) + \gamma(B) - \gamma(\{(0, 0)\}) = 1 + 1 - 0 = 2, \end{aligned}$$

which is a contradiction. Hence, we have $\mathbb{R} \times \{0\} \subseteq B$. Because H is the intersection of all the full measure subspaces and we have just seen that these all contain $\mathbb{R} \times \{0\}$, we conclude $\mathbb{R} \times \{0\} \subseteq H$.

In total, we have $H = \mathbb{R} \times \{0\}$.

Exercise 3.4

Let γ, μ be equivalent Gaussian measures on a Banach space X. We want to show that the Cameron-Martin spaces H_γ and H_μ with respect to γ and μ respectively coincide. We start by noting that $\gamma \approx \mu$ implies $\gamma_x \approx \mu_x$ for any $x \in X$ and that the relation $\approx$ for measures is an equivalence relation. Let now $h \in H_\gamma$. By Theorem 3.1.5 this means $\gamma_h \approx \gamma$. Together with $\gamma \approx \mu$ and $\gamma_h \approx \mu_h$ this implies $\mu \approx \mu_h$. Hence $h \in H_\mu$. Using the same argumentation for a given $h \in H_\mu$, we get $h \in H_\gamma$ and hence $H_\gamma = H_\mu$.

Let γ, μ be equivalent centered Gaussian measures on a separable Banach space X. We want to show that $X^*_\gamma = X^*_\mu$. Choose an arbitrary $f \in X^*_\gamma$. By the definiton of X^*_γ there is a sequence $(f_n)_{n\in\mathbb{N}}$ in X^* such that $(j_\gamma(f_n))_{n\in\mathbb{N}}$ converges to f in $L^2(X,\gamma)$. Here,

$$j_\nu : X^* \to L^2(X,\nu), \quad f \mapsto f - a_\nu(f)$$

for any Gaussian measure ν. As γ and μ are centered, the functions $j_\gamma(f_n)$ and f_n (and $j_\mu(f_n)$) coincide as functions from X to $\mathbb{R}$. Hence, we will simply write f_n instead of $j_\gamma(f_n)$ (or $j_\mu(f_n)$). Due to the convergence of $(f_n)_{n\in\mathbb{N}}$ in $L^2(X,\gamma)$, there is a subsequence of $(f_n)_{n\in\mathbb{N}}$ that converges γ-almost everywhere to f. Without loss of generality, that subsequence is $(f_n)_{n\in\mathbb{N}}$ itself. By the equivalence of γ and μ, the sequence $(f_n)_{n\in\mathbb{N}}$ converges μ-almost everywhere to f. This already implies the convergence of f_n to f in measure with respect to μ. In order to prove this, let $\epsilon > 0$ and set $M_k^\epsilon := \{x \in X : |f_n(x) - f(x)| > \epsilon\}$. Due to

$$\frac{1+\epsilon}{\epsilon} \cdot \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} > 1$$

for each $x \in M_k^\epsilon$ and the theorem of dominated convergence, we get the assertion by

$$\begin{aligned}
\mu(M_k^\epsilon) = \int_{M_k^\epsilon} \mu(\mathrm{d}x) &\le \int_{M_k^\epsilon} \frac{1+\epsilon}{\epsilon} \cdot \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \mu(\mathrm{d}x) \\
&\le \frac{1+\epsilon}{\epsilon} \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \mu(\mathrm{d}x) \to 0
\end{aligned}$$

for $n \to \infty$. As we now have the convergence in measure of f_n to f with respect to μ, it follows from Theorem 8.3.8 that $f \in X_\mu^*$, i.e. $X_\gamma^* \subseteq X_\mu^*$. By the same argumentation we get the reverse inclusion and hence $X_\gamma^* = X_\mu^*$.

Let X be a separable Banach space and γ, μ be centered Gaussian measures on X with $\gamma \perp \mu$. We show that then $\gamma_x \perp \mu_y$ for any $x, y \in X$. Let us first notice that in our case

$$\gamma_x \perp \mu_y \text{ for any } x, y \in X \Longleftrightarrow \gamma_x \perp \mu \text{ for any } x \in X.$$

The implication from the left to right hand side is easy. In order to prove the other implication let $x, y \in X$ be arbitrary and set $z := x - y$. By our assumption, there is a set $E \subset X$ such that $\gamma_z(E) = 0$ and $\mu(E^C) = 0$. Set $F := E + y$. Then $\gamma_x(F) = \gamma_z(E) = 0$ and

$$\mu_y(F^C) = \mu(F^C - y) = \mu((F - y)^C) = \mu(E^C) = 0,$$

i.e. $\gamma_x \perp \mu_y$. Hence, we only have to show the right hand side of the equivalence above. Let $x \in X$. If $x \in H_\gamma$, then $\gamma_x \approx \gamma \perp \mu$ and therefore $\gamma_x \perp \mu$. If $x \notin H_\gamma$, then by Proposition 3.1.8 (ii) there is a linear subspace $S \subseteq X$ such that $\gamma(S) = 1$ and $x \notin S$. We will see that $S + x$ is a set we can use to show the mutual singularity of γ_x and μ. At first we have $\gamma_x(S + x) = \gamma(S) = 1$. Secondly, $\mu(S+x) = \mu(S-x)$ due to the symmetry of μ. As $S+x$ and $S-x$ are disjoint, this implies $\mu(S + x) \leq 1/2$. This already means $\mu(S + x) = 0$ by Theorem 8.2.2 and therefore $\mu((S + x)^C) = 1$. Hence $\gamma_x \perp \mu$.

Exercise 3.5

We consider a Gaussian measure γ on the space X. For $x \in X$ the measure γ_x is given by

$$\gamma_x(B) = \gamma(B - x), \quad B \in \mathcal{B}(X).$$

For the proof that γ and γ_x have the same Cameron-Martin space, we will use the characterization given by the Cameron-Martin Theorem (Theorem 3.1.5).

Let H and H_x be the Cameron-Martin spaces of γ and γ_x, respectively. For $h \in X$ the Cameron-Martin Theorem yields that $h \in H$ if and only if $\gamma_h \approx \gamma$. But by definition of the equivalence of two measures, this means that

$$\gamma_h(B) = 0 \Leftrightarrow \gamma(B) = 0 \text{ for all } B \in \mathcal{B}(X),$$

which can also be formulated as

$$\gamma(B - h - x) = 0 \Leftrightarrow \gamma(B - x) = 0 \text{ for all } B \in \mathcal{B}(X)$$

or, expressed differently,

$$(\gamma_x)_h(B) = 0 \Leftrightarrow \gamma_x(B) = 0 \text{ for all } B \in \mathcal{B}(X).$$

But the last assertion is actually the definition of $(\gamma_x)_h \approx \gamma_x$, which is equivalent to $h \in H_x$ by the Cameron-Martin Theorem.

All in all, we obtain that the sets H and H_x coincide, i.e. γ_x has the same Cameron-Martin space as γ.