Solutions of exercises: Lecture 2

Marrakesh team

Exercise 1

Let γ be a Gaussian measure (but not Dirac) on a Banach space X and let us fix $x \in X$ and $r > 0$. Since γ is not Dirac, there is a functional f ($0 \neq f \in X^*$) such that $\gamma \circ f^{-1}$ is nondegenerate, then there exists $M > 0$ such that

$$\overline{B}(x,r) \subset f^{-1}([-M, M])$$

which implies that

$$\gamma(\overline{B}(x,r)) \leq \gamma \circ f^{-1}([-M, M]).$$

Since

$$\gamma \circ f^{-1}([-M, M]) < 1$$

we have

$$\gamma(B(x,r)) < 1$$

which prove the result.

Exercise 2

Let X be an infinite dimensional Banach space. As suggested in the hint, we will construct a sequence of elements in the unit sphere such that, for every $m, n \geq 1$

$$m \neq n \qquad \text{implies} \qquad \|e_m - e_n\| \geq \frac{1}{2}.$$

For that we use Riesz's lemma, see [Br, lemma 6.1].

Let $e_1 \in X \setminus \{0\}$ with $\|e_1\| = 1$ and $Y_1 := \text{span}(e_1)$. The subspace Y_1 is closed and proper in X. We have, by Riesz's lemma, there exists $e_2 \in X$ such that $\|e_2\| = 1$ and $\|e_1 - e_2\| \geq d(e_2, Y_1) \geq \frac{1}{2}$. We use the same argument to construct a sequence $(e_n)_{n\geq 1}$ such that

$$\begin{cases} \forall\, n \geq 1, \|e_n\| = 1 \\ \text{and} \\ \forall n, m \geq 1,\, n \neq m \text{ implies } \|e_n - e_m\| \geq \dfrac{1}{2}. \end{cases}$$

For every $n \geq 1$ we consider the balls B_n with centre $4re_n$ and radius $r > 0$. The balls B_n are pairwise disjoint and by assumption they have the same measure say $\mu(B_n) = \alpha > 0$ for every $n \geq 1$. Thus, we have

$$\cup_{n\geq 1} B_n \subset B(0, 5r) \Longrightarrow \mu(B(0,5r)) \geq \sum_{n=1}^{+\infty} \mu(B_n) = \sum_{n=1}^{+\infty} \alpha = +\infty,$$

finally, we have $\mu(A) = +\infty$ for every open set A. This proves Proposition 2.2.1 in the case of a Banach space.

Exercise 3

Let γ be a centred Gaussian measure on a separable Banach space X, we prove that
(i) γ is degenerate if and only if there exists $0 \neq f \in X^*$ such that $\hat{\gamma}(f) = 1$.
We assume that γ is degenerate, then there exists $0 \neq f \in X^*$ such that $\gamma \circ f^{-1} = \delta_0$.
Indeed

$$\begin{aligned}\hat{\gamma}(f) &= \int_X e^{if(x)}\gamma(dx) \\ &= \int_{\mathbb{R}} e^{it}\gamma \circ f^{-1}(dt) \\ &= \int_{\mathbb{R}} e^{it}\delta_0(dt) \\ &= 1.\end{aligned}$$

Reciprocally, if there exists $0 \neq f \in X^*$ such that $\hat{\gamma}(f) = 1$, then by Proposition 2.1.2 we have

$$1 = \hat{\gamma}(f) = e^{B(f,f)}$$

which implies that

$$\sigma^2 = B(f, f) = 0$$

which means that γ is a degenerate Gaussian measure on X.
(ii) There exists $0 \neq f \in X^*$ such that $\hat{\gamma}(f) = 1$ if and only if there exists a proper closed subspace $V \subset X$ with $\gamma(V) = 1$.
If there exists $0 \neq f \in X^*$ such that $\hat{\gamma}(f) = 1$, then the subspace $V = f^{-1}(\{0\})$ is proper in X (if it is not proper, then necessarily $f = 0$ which contradicts our hypothesis) and we have that

$$\begin{aligned}\gamma(V) &= \gamma(f^{-1}(\{0\})) \\ &= \gamma \circ f^{-1}(\{0\}) \\ &= \delta_0(\{0\}) \\ &= 1.\end{aligned}$$

Reciprocally, if there exists a proper closed subspace $V \subset X$ with $\gamma(V) = 1$, then by Hahn−Banach theorem (geometrical form) it follows that, for fixed $x \notin V$ there exists $0 \neq f \in X^*$ such that $V \subset \ker(f)$ and $f(x) = 1$. Then $\gamma \circ f^{-1} = \delta_0$. This proves the result.

Exercise 4
Let γ be a centred Gaussian measure on a Banach space X. For any choice $f_1, ..., f_d$ in X^*, we set

$$P : X \longrightarrow \mathbb{R}^d,\ x \longmapsto P(x) = (f_1(x), ..., f_d(x)).$$

(i) For every $\xi \in \mathbb{R}^d$, we have

$$\begin{aligned}\widehat{\gamma \circ P^{-1}}(\xi) &= \int_{\mathbb{R}^n} e^{i\xi \cdot y}\gamma \circ P^{-1}(dy) \\ &= \int_X e^{i\xi \cdot P(x)}\gamma(dx) \\ &= \int_X e^{iP^*(\xi)\cdot x}\gamma(dx) \\ &= \hat{\gamma}(P^*\xi)\end{aligned}$$

where

$$P^* : \mathbb{R}^d \longrightarrow X^*, \xi \longmapsto P^*(\xi) = \sum_{i=1}^{d} \xi_i f_i.$$

On the other hand

$$\hat{\gamma}(P^*(\xi)) \;=\; e^{ia(P^*(\xi)) - \frac{1}{2}B(P^*(\xi), P^*(\xi))}$$

since γ is centred, we have $a(P^*(\xi)) = \displaystyle\int_X P^*(\xi) \cdot \gamma(dx) = \sum_{i=1}^{d} \xi_i \int_X f_i(x)\gamma(dx) = 0$

and $B(P^*(\xi), P^*(\xi)) =< P^*(\xi), P^*(\xi) >_{L^2(X,\gamma)} = \displaystyle\sum_{i=1}^{d}\sum_{j=1}^{d} \xi_i\xi_j < f_i, f_j >_{L^2(X,\gamma)} = Q\xi \cdot \xi$

where $Q_{i,j} =< f_i, f_j >_{L^2(X,\gamma)}$, then $\gamma \circ P^{-1}$ is the Gaussian probability measure on $\mathbb{R}^d$ $\mathcal{N}(0, Q)$.

(ii) Let $L : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be a linear map. For every $\xi \in \mathbb{R}^d$, we have

$$\begin{aligned}
\gamma \circ \widehat{(L \circ P)}^{-1}(\xi) &= \int_{\mathbb{R}^d} e^{i\xi\cdot y} \gamma \circ (L \circ P)^{-1}(dy) \\
&= \int_X e^{i\xi\cdot(L\circ P)(x)} \gamma(dx) \\
&= \int_X e^{i(P^*\circ L^*)(\xi)\cdot x} \gamma(dx) \\
&= \hat{\gamma}((P^* \circ L^*)(\xi))
\end{aligned}$$

where $L^* : \mathbb{R}^n \longrightarrow \mathbb{R}^d$, $\xi := (\xi_1, ..., \xi_n) \longmapsto (y_1, ..., y_d) =: y$, and then $P^* \circ L^* : \mathbb{R}^n \longrightarrow X^*$ defined by $P^* \circ L^*(\xi) = \displaystyle\sum_{i=1}^{d} y_i f_i$.

Since γ is centred, we have $a((P^* \circ L^*)\xi) = \displaystyle\int_X (P^* \circ L^*)\xi(x)\gamma(dx) = \sum_{i=1}^{d} y_i \int_X f_i(x)\gamma(dx) = 0$

and

$$\begin{aligned}
B((P^* \circ L^*)\xi, (P^* \circ L^*)\xi) &= < (P^* \circ L^*)\xi, (P^* \circ L^*)\xi >_{L^2(X,\gamma)} \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d} y_i y_j < f_i, f_j >_{L^2(X,\gamma)} \\
&= LQL^*\xi \cdot \xi,
\end{aligned}$$

then $\gamma \circ (L \circ P)^{-1}$ is the Gaussian probability measure on $\mathbb{R}^n$ $\mathcal{N}(0, LQL^*)$.

Exercise 5

Let $\gamma \sim \mathcal{N}(0, Q)$ be a nondegenerate centred Gaussian probability measure on $\mathbb{R}^d$. As suggested in the hint, let us consider the function $F(\varepsilon) = \displaystyle\int_{\mathbb{R}^d} e^{\varepsilon|x|^2}\gamma(dx)$. Note that, by the Fernique Thoerem it follows that, there exists $\alpha > 0$ (it will be specified later) such that the integral defined by F is finite for every $0 \leq \varepsilon < \alpha$ and by the theorem of differentiation under the integral sign we prove that F is of classe C^2. Since γ is nondegenerate it follows by Proposition 1.2.4 that

$$F(\varepsilon) = \frac{1}{\sqrt{(2\pi)^d \det(Q)}} \int_{\mathbb{R}^d} \exp\{\varepsilon \mid x \mid^2 - \frac{1}{2}(Q^{-1}x \cdot x)\} dx.$$

Moreover, there exists an orthogonal matrix P such that $P^*QP = D$ and $P^*P = PP^* = I$, where $D := \mathrm{diag}(\lambda_1, ..., \lambda_n)$. Note that $\mathrm{Tr}(Q) = \sum_{i=1}^{d} \lambda_i$ and $\det(Q) = \prod_{i=1}^{d} \lambda_i$, then, for every $0 \leq \varepsilon < \alpha < \frac{1}{2} \max\{\lambda_1, ..., \lambda_n\}$

$$\begin{aligned}
F(\varepsilon) &= \frac{1}{\sqrt{(2\pi)^d \det(Q)}} \int_{\mathbb{R}^d} \exp\{\varepsilon(Py \cdot Py) - \frac{1}{2}(Q^{-1}Py \cdot Py)\} dy \qquad \text{by change of variable } x = Py \\
&= \frac{1}{\sqrt{(2\pi)^d \det(Q)}} \int_{\mathbb{R}^d} \exp\{\varepsilon(y \cdot y) - \frac{1}{2}(D^{-1}y \cdot y)\} dy \\
&= \frac{1}{\sqrt{(2\pi)^d \det(Q)}} \int_{\mathbb{R}^d} \exp\{\varepsilon \sum_{i=1}^{d} y_i^2 - \frac{1}{2} \sum_{i=1}^{d} \frac{y_i^2}{\lambda_i}\} dy \\
&= \frac{1}{\sqrt{(2\pi)^d \det(Q)}} \int_{\mathbb{R}^d} \exp\{\sum_{i=1}^{d} \left[\varepsilon - \frac{1}{2\lambda_i}\right] y_i^2\} dy \\
&= \frac{1}{\sqrt{\det(Q)}} \prod_{i=1}^{d} \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} \exp\{\left[\varepsilon - \frac{1}{2\lambda_i}\right] y_i^2\} dy_i,
\end{aligned}$$

since

$$\frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} \exp\{\left[\varepsilon - \frac{1}{2\lambda_i}\right] y_i^2\} dy_i = \left(\frac{\lambda_i}{1 - 2\lambda_i\varepsilon}\right)^{\frac{1}{2}},$$

then

$$F(\varepsilon) = \prod_{i=1}^{d} \left(\frac{1}{1 - 2\lambda_i\varepsilon}\right)^{\frac{1}{2}},$$

so

$$F^{'}(\varepsilon) = \sum_{i=1}^{d} \frac{\lambda_i}{(1 - 2\lambda_i\varepsilon)^{\frac{3}{2}}} \prod_{j=1, j\neq i}^{d} \left(\frac{1}{1 - 2\lambda_j\varepsilon}\right)^{\frac{1}{2}} = \sum_{i=1}^{d} \frac{\lambda_i}{1 - 2\lambda_i\varepsilon} F(\varepsilon),$$

and

$$F^{''}(\varepsilon) = 2\sum_{i=1}^{d} \frac{\lambda_i^2}{(1 - 2\lambda_i\varepsilon)^2} F(\varepsilon) + \sum_{i=1}^{d} \frac{\lambda_i}{(1 - 2\lambda_i\varepsilon)} F^{'}(\varepsilon),$$

thus

$$\begin{aligned}
F^{''}(0) &= 2\sum_{i=1}^{d} \lambda_i^2 F(0) + \sum_{i=1}^{d} \lambda_i F^{'}(0) \\
&= 2\sum_{i=1}^{d} \lambda_i^2 + \sum_{i=1}^{d} \lambda_i \sum_{j=1}^{d} \lambda_j \\
&= 2\mathrm{Tr}(Q^2) + (\mathrm{Tr}(Q))^2.
\end{aligned}$$

It follows,

$$F^{''}(0) = \int_{\mathbb{R}^d} \mid x \mid^4 \gamma(dx).$$

This proves the result.

Remark: In the case of γ is not centred, we can't find this result (it suffices to verify in the case when $d = 1$).

Exercise 6

Let γ be a centred Gaussian measure on a separable Banach space X and let $f \in X^*$. We denote by $\sigma^2 = B_\gamma(f, f)$

(i) $I = \displaystyle\int_X e^{f(x)}\gamma(dx)$

$$\begin{aligned}
\int_X e^{f(x)}\gamma(dx) &= \int_{\mathbb{R}} e^t \gamma \circ f^{-1}(dt) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{\frac{-1}{2\sigma^2}(t^2 - 2\sigma^2 t)} dt \\
&= \frac{e^{\frac{\sigma^2}{2}}}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\left(\frac{t-\sigma^2}{\sqrt{2}\sigma}\right)^2} dt \\
&= e^{\frac{\sigma^2}{2}}
\end{aligned}$$

then

$$I = e^{\frac{\sigma^2}{2}}.$$

(ii) $I_k = \displaystyle\int_X (f(x))^k \gamma(dx) \quad \text{for } k \in \mathbb{N}$

$$\begin{aligned}
\int_X (f(x))^k \gamma(dx) &= \int_{\mathbb{R}} t^k \gamma \circ f^{-1}(dt) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} t^k e^{\frac{-t^2}{2\sigma^2}} dt
\end{aligned}$$

the function $t \longmapsto e^{\frac{-t^2}{2\sigma^2}}$ is even and $t \longmapsto t^k$ is odd if k is odd, in this case $I_k = 0$. Now let $k = 2p$ where $p \in \mathbb{N}$, then

$$\begin{aligned}
I_k &= \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} t^{2p} e^{\frac{-t^2}{2\sigma^2}} dt \\
&= \frac{2^p \sigma^{2p}}{\sqrt{\pi}} \int_0^{+\infty} s^{p-\frac{1}{2}} e^{-s} ds \\
&= \frac{2^p \sigma^{2p}}{\sqrt{\pi}} \Gamma(p + \frac{1}{2}) \\
&= \frac{2^p \sigma^{2p} (2p)! \sqrt{\pi}}{\sqrt{\pi} 2^{2p} p!}
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function, then we have finally

$$I_k = \frac{\sigma^k k!}{2^{\frac{k}{2}} (\frac{k}{2})!}.$$