Lecture 2

Gaussian measures in infinite dimension

In this lecture, after recalling a few notions about σ-algebras in Banach spaces, we introduce Gaussian measures in separable Banach spaces and we prove the Fernique Theorem. This is a powerful tool to get precise estimates on the Gaussian integrals. In most of the course, our framework is an infinite dimensional separable real Banach space that we denote by X, with norm $\|\cdot\|$ or $\|\cdot\|_X$ when there is risk of confusion. The open (resp. closed) ball with centre $x \in X$ and radius $r > 0$ will be denoted by $B(x, r)$ (resp. $\overline{B}(x, r)$). We denote by X^*, with norm $\|\cdot\|_{X^*}$, the topological dual of X consisting of all linear continuous functions $f : X \to \mathbb{R}$. Sometimes, we shall discuss the case of a separable Hilbert space in order to highlight some special features. The only example that is not a Banach space is $\mathbb{R}^I$ (see Lecture 3), but we prefer to describe this as a particular case rather than to present a more general abstract theory.

2.1 σ-algebras in infinite dimensional spaces and characteristic functions

In order to present further properties of measures in X, and in particular approximation of measures and functions, it is useful to start with a discussion on the relevant underlying σ-algebras. Besides the Borel σ-algebra, to take advantage of the finite dimensional reductions we shall frequently encounter in the sequel, we introduce the σ-algebra $\mathscr{E}(X)$ generated by the *cylindrical* sets, i.e, the sets of the form

$$C = \Big\{x \in X : \ (f_1(x), \ldots, f_n(x)) \in C_0\Big\},$$

where $f_1, \ldots, f_n \in X^*$ and $C_0 \in \mathscr{B}(\mathbb{R}^n)$, called a *base* of C. According to Definition 1.1.8, $\mathscr{E}(X) = \mathscr{E}(X, X^*)$. The following important result holds. We do not present its most general version, but we do not even confine ourselves to Banach spaces because we shall apply it to $\mathbb{R}^\infty$, which is a Fréchet space. We recall that a Fréchet space is a complete metrisable locally convex topological vector space, i.e., a vector space endowed with a sequence of seminorms that generate a metrisable topology such that the space is complete.

Theorem 2.1.1. *If X is a separable Fréchet space, then $\mathscr{E}(X) = \mathscr{B}(X)$. Moreover, there is a countable family $F \subset X^*$ separating the points in X (i.e., such that for every pair of points $x \neq y \in X$ there is $f \in F$ such that $f(x) \neq f(y)$) such that $\mathscr{E}(X) = \mathscr{E}(X, F)$.*

Proof. Let (x_n) be a sequence dense in X, and denote by (p_k) a family of seminorms which defines the topology of X. By the Hahn-Banach theorem for every n and k there is $\ell_{n,k} \in X^*$ such that $p_k(x_n) = \ell_{n,k}(x_n)$ and $\sup\{\ell_{n,k}(x) : \ p_k(x) \leq 1\} = 1$. As a consequence, for every $x \in X$ and $k \in \mathbb{N}$ we have $p_k(x) = \sup_n\{\ell_{n,k}(x)\}$. Therefore, for every $r > 0$

$$\overline{B}_k(x, r) := \{y \in X : \ p_k(y - x) \leq r\} = \bigcap_{n\in\mathbb{N}} \{y \in X : \ \ell_{n,k}(y - x) \leq r\} \in \mathscr{E}(X).$$

As X is separable, there is a countable base of its topology. For instance,we may take the sets $B_k(x_n, r)$ with rational r, see [DS1, I.6.2 and I.6.12]. These sets are in $\mathscr{E}(X)$, hence the inclusion $\mathscr{B}(X) \subset \mathscr{E}(X)$ holds. The converse inclusion is trivial.
To prove the last statement, just take $F = \{\ell_{n,k}, \ n, k \in \mathbb{N}\}$. It is obviously a countable family; let us show that it separates points. If $x \neq y$, there is $k \in \mathbb{N}$ such that $p_k(x - y) = \sup_n \ell_{n,k}(x - y) > 0$ and therefore there is $\bar{n} \in \mathbb{N}$ such that $\ell_{\bar{n},k}(x - y) > 0$. $\square$

In the discussion of the properties of Gaussian measures in Banach spaces, as in $\mathbb{R}^d$, the characteristic functions, defined by

$$\hat{\mu}(f) := \int_X \exp\{if(x)\}\, \mu(dx), \qquad f \in X^*,$$

play an important role. The properties of characteristic functions seen in Lecture 1 can be extended to the present context. We discuss in detail only the extension of property (iii) (the injectivity), which is the most important for our purposes. In the following proposition, we use the coincidence criterion for measures agreeing on a system of generators of the σ-algebra, see e.g. [D, Theorem 3.1.10].

Proposition 2.1.2. *Let $\mu_1, \ \mu_2$ be two probability measures on $(X, \mathscr{B}(X))$. If $\widehat{\mu}_1 = \widehat{\mu}_2$ then $\mu_1 = \mu_2$.*

Proof. It is enough to show that if $\hat{\mu} = 0$ then $\mu = 0$ and in particular, by Theorem 2.1.1, that $\mu(C) = 0$ when C is a cylinder with base $C_0 \in \mathscr{B}(\mathbb{R}^d)$. Let be $\hat{\mu} = 0$, consider $F = \operatorname{span}\{f_1, \ldots, f_d\} \subset X^*$ and define $\mu_F = \mu \circ P_F^{-1}$, where $P_F : X \to \mathbb{R}^d$ is given by $P_F(x) = (f_1(x), \ldots, f_d(x))$. Then for any $\xi \in \mathbb{R}^d$

$$\widehat{\mu}_F(\xi) = \int_F \exp\{i\xi \cdot y\}\, \mu_F(dy) = \int_X \exp\{i\xi \cdot P_F(x)\}\, \mu(dx) = \int_X \exp\{iP_F^*\xi(x)\}\, \mu(dx) = 0,$$

where $P_F^* : \mathbb{R}^d \to X^*$ is the adjoint map

$$P_F^*(\xi) = \sum_{i=1}^{d} \xi_i f_i.$$

It follows that $\mu_F = 0$ and therefore the restriction of μ to the σ-algebra $\mathscr{E}(X, F)$ is the null measure. $\square$

2.2 Gaussian measures in infinite dimensional spaces

Measure theory in infinite dimensional spaces is far from being a trivial issue, because there is no equivalent of the Lebesgue measure, i.e., there is no nontrivial measure invariant by translations.

Proposition 2.2.1. *Let X be a separable Hilbert space. If $\mu : \mathscr{B}(X) \to [0, +\infty]$ is a σ-additive set function such that:*

(i) $\mu(x + B) = \mu(B)$ for every $x \in X$, $B \in \mathscr{B}(X)$,

(ii) $\mu(B(0, r)) > 0$ for every $r > 0$,

then $\mu(A) = +\infty$ for every open set A.

Proof. Assume that μ satisfies (i) and (ii), and let $\{e_n\}$ be an orthonormal basis in X. For any $n \in \mathbb{N}$ consider the balls B_n with centre $2re_n$ and radius $r > 0$; they are pairwise disjoint and by assumption they have the same measure, say $\mu(B_n) = m > 0$ for all $n \in \mathbb{N}$. Then,

$$\bigcup_{n=1}^{\infty} B_n \subset B(0, 3r) \quad \Longrightarrow \quad \mu(B(0, 3r)) \geq \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} m = +\infty,$$

hence $\mu(A) = +\infty$ for every open set A. $\square$

Definition 2.2.2 (Gaussian measures on X)**.** *Let X be a Banach space. A probability measure γ on $(X, \mathscr{B}(X))$ is said to be Gaussian if $\gamma \circ f^{-1}$ is a Gaussian measure in $\mathbb{R}$ for every $f \in X^*$. The measure γ is called* centred *(or* symmetric*) if all the measures $\gamma \circ f^{-1}$ are centred and it is called* nondegenerate *if for any $f \neq 0$ the measure $\gamma \circ f^{-1}$ is nondegenerate.*

Our first task is to give characterisations of Gaussian measures in infinite dimensions in terms of characteristic functions analogous to those seen in $\mathbb{R}^d$, Proposition 1.2.4.

Notice that if $f \in X^*$ then $f \in L^p(X, \gamma)$ for every $p \geq 1$: indeed, the integral

$$\int_X |f(x)|^p \, \gamma(dx) = \int_{\mathbb{R}} |t|^p \, (\gamma \circ f^{-1})(dt)$$

is finite because $\gamma \circ f^{-1}$ is Gaussian in $\mathbb{R}$. Therefore, we can give the following definition.

Definition 2.2.3. *We define the* mean a_γ *and the* covariance B_γ *of* γ *by*

$$a_\gamma(f) := \int_X f(x)\,\gamma(dx), \qquad f \in X^*, \tag{2.2.1}$$

$$B_\gamma(f,g) := \int_X [f(x) - a_\gamma(f)]\,[g(x) - a_\gamma(g)]\,\gamma(dx), \qquad f, g \in X^*. \tag{2.2.2}$$

Observe that $f \mapsto a_\gamma(f)$ is linear and $(f,g) \mapsto B_\gamma(f,g)$ is bilinear in X^*. Moreover, $B_\gamma(f,f) = \|f - a_\gamma(f)\|^2_{L^2(X,\gamma)} \geq 0$ for every $f \in X^*$.

Theorem 2.2.4. *A Borel probability measure* γ *on* X *is Gaussian if and only if its characteristic function is given by*

$$\hat{\gamma}(f) = \exp\Big\{ia(f) - \frac{1}{2}B(f,f)\Big\}, \qquad f \in X^*, \tag{2.2.3}$$

where a *is a linear functional on* X^* *and* B *is a nonnegative symmetric bilinear form on* X^**.*

Proof. Assume that γ is Gaussian. Let us show that $\hat{\gamma}$ is given by (2.2.3) with $a = a_\gamma$ and $B = B_\gamma$. Indeed, we have:

$$\widehat{\gamma}(f) = \int_{\mathbb{R}} \exp\{i\xi\}(\gamma \circ f^{-1})(d\xi) = \exp\Big\{im - \frac{1}{2}\sigma^2\Big\},$$

where m and σ^2 are the mean and the covariance of $\gamma \circ f^{-1}$, given by

$$m = \int_{\mathbb{R}} \xi(\gamma \circ f^{-1})(d\xi) = \int_X f(x)\gamma(dx) = a_\gamma(f),$$

and

$$\sigma^2 = \int_{\mathbb{R}} (\xi - m)^2(\gamma \circ f^{-1})(d\xi) = \int_X (f(x) - a_\gamma(f))^2\gamma(dx) = B_\gamma(f,f).$$

Conversely, let γ be a Borel probability measure on X and assume that (2.2.3) holds. Since a is linear and B is bilinear, we can compute the Fourier transform of $\gamma \circ f^{-1}$, for $f \in X^*$, as follows:

$$\begin{aligned}\widehat{\gamma \circ f^{-1}}(\tau) &= \int_{\mathbb{R}} \exp\{i\tau t\}\,(\gamma \circ f^{-1})(dt) = \int_X \exp\{i\tau f(x)\}\,\gamma(dx) \\ &= \exp\Big\{i\tau a(f) - \frac{1}{2}\tau^2 B(f,f)\Big\}.\end{aligned}$$

According to Remark 1.2.2, $\gamma \circ f^{-1} = \mathscr{N}(a(f), B(f,f))$ is Gaussian and we are done. $\square$

Remark 2.2.5. We point out that at the moment we have proved that $a_\gamma \in (X^*)'$ and $B_\gamma \in (X^* \times X^*)'$, the algebraic duals consisting of linear functions (not necessarily continuous). We shall see that a_γ and B_γ are in fact continuous.

As in the finite dimensional case, we say that γ is *centred* if $a_\gamma = 0$; in this case, the bilinear form B_γ is nothing but the restriction of the inner product in $L^2(X,\gamma)$ to X^*,

$$B_\gamma(f,g) = \int_X f(x)g(x)\,\gamma(dx), \qquad B_\gamma(f,f) = \|f\|^2_{L^2(X,\gamma)}. \tag{2.2.4}$$

In the sequel we shall frequently consider centred Gaussian measures; this requirement is equivalent to the following symmetry property.

Proposition 2.2.6. *Let γ be a Gaussian measure on a Banach space X and define the measure μ by*

$$\mu(B) := \gamma(-B), \qquad \forall B \in \mathscr{B}(X).$$

Then, γ is centred if and only if $\gamma = \mu$.

Proof. We know that $\widehat{\gamma}(f) = \exp\{ia_\gamma(f) - \frac{1}{2}\|f - a_\gamma(f)\|^2_{L^2(X,\gamma)}\}$. On the other hand, since $\mu = \gamma \circ R^{-1}$ with $R : X \to X$ given by $R(x) = -x$, then

$$\widehat{\mu}(f) = \int_X e^{if(x)}\mu(dx) = \int_X e^{-if(x)}\gamma(dx) = \exp\{-ia_\gamma(f) - \frac{1}{2}\|f - a_\gamma(f)\|^2_{L^2(X,\gamma)}\}.$$

Then $\widehat{\mu} = \widehat{\gamma}$ if and only if $a_\gamma(f) = 0$ for any $f \in X^*$, whence the statement follows by Proposition 2.1.2. $\square$

Let us draw some interesting (and useful) consequences from the above result.

Proposition 2.2.7. *Let γ be a centred Gaussian measure on X.*

(i) If μ is a centred Gaussian measure on a separable Banach space Y, then $\gamma \otimes \mu$ is a centred Gaussian measure on $X \times Y$.

(ii) If μ is another centred Gaussian measure on X, then the convolution measure $\gamma * \mu$, defined as the image measure in X of $\gamma \otimes \mu$ on $X \times X$ under the map $(x,y) \mapsto x + y$ is a centred Gaussian measure and is given by

$$\gamma * \mu(B) = \int_X \mu(B - x)\gamma(dx) = \int_X \gamma(B - x)\mu(dx). \tag{2.2.5}$$

(iii) For every $\theta \in \mathbb{R}$ the image measure $(\gamma \otimes \gamma) \circ R_\theta^{-1}$ in $X \times X$ under the map $R_\theta : X \times X \to X \times X$, $R_\theta(x,y) := (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ is again $\gamma \otimes \gamma$.

(iv) For every $\theta \in \mathbb{R}$ the image measures $(\gamma \otimes \gamma) \circ \phi_i^{-1}$, $i = 1, 2$ in X under the maps $\phi_i : X \times X \to X$,

$$\phi_1(x,y) := x\cos\theta + y\sin\theta, \quad \phi_2(x,y) := -x\sin\theta + y\cos\theta$$

are again γ.

Proof. All these results follow quite easily by computing the relevant characteristic functions. We start with (i) taking into account that for $f \in X^*$ and $g \in Y^*$ we have

$$B_\gamma(f,f) = \|f\|^2_{L^2(X,\gamma)}, \qquad B_\mu(g,g) = \|g\|^2_{L^2(Y,\mu)}.$$

For every $\ell \in (X \times Y)^*$, we define f and g by

$$\ell(x,y) = \ell(x,0) + \ell(0,y) =: f(x) + g(y).$$

Then

$$\begin{aligned}
\widehat{\gamma \otimes \mu}(\ell) &= \int_{X\times Y} \exp\{i\ell(x,y)\}(\gamma\otimes\mu)(d(x,y)) \\
&= \int_X \exp\{if(x)\}\gamma(dx) \int_Y \exp\{ig(y)\}\mu(dy) \\
&= \exp\left\{-\frac{1}{2}\|f\|^2_{L^2(X,\gamma)} - \frac{1}{2}\|g\|^2_{L^2(Y,\mu)}\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\|f\|^2_{L^2(X,\gamma)} + \|g\|^2_{L^2(Y,\mu)}\right)\right\}.
\end{aligned}$$

On the other hand, since γ and μ are centred

$$\int_{X\times Y} f(x)g(y)(\gamma\otimes\mu)(d(x,y)) = \int_X f(x)\gamma(dx)\int_Y g(y)\mu(dy) = 0,$$

and therefore

$$\begin{aligned}
\|f\|^2_{L^2(X,\gamma)} + \|g\|^2_{L^2(Y,\mu)} &= \int_X f(x)^2\gamma(dx) + \int_Y g(y)^2\mu(dy) \\
&= \int_X \ell(x,0)^2\gamma(dx) + \int_Y \ell(0,y)^2\mu(dy) \\
&= \int_{X\times Y} \ell(x,0)^2(\gamma\otimes\mu)(d(x,y)) + \int_{X\times Y} \ell(0,y)^2(\gamma\otimes\mu)(d(x,y)) \\
&= \int_{X\times Y} (\ell(x,0)^2 + \ell(0,y)^2)(\gamma\otimes\mu)(d(x,y)) \\
&= \int_{X\times Y} (\ell(x,0) + \ell(0,y))^2(\gamma\otimes\mu)(d(x,y)) \\
&= \int_{X\times Y} \ell(x,y)^2(\gamma\otimes\mu)(d(x,y)).
\end{aligned}$$

So we have

$$B_{\gamma\otimes\mu}(\ell,\ell) = B_\gamma(f,f) + B_\mu(g,g)$$

if we decompose ℓ by $\ell(x,y) = f(x) + g(y)$, $f(x) = \ell(x,0)$, $g(y) = \ell(0,y)$ as before.

The proof of statement (ii) is similar; indeed if $h : X \times X \to X$ is given by $h(x,y) = x + y$, then

$$\begin{aligned}\widehat{(\gamma \otimes \mu) \circ h^{-1}}(\ell) &= \int_X \exp\{i\ell(x)\} \left((\gamma \otimes \mu) \circ h^{-1}\right)(dx) \\ &= \int_{X \times Y} \exp\{i\ell(h(x,y))\}(\gamma \otimes \mu)(d(x,y)) \\ &= \int_X \exp\{i\ell(x)\}\gamma(dx) \int_Y \exp\{i\ell(y)\}\mu(dy) \\ &= \exp\left\{-\frac{1}{2}\left(\|\ell\|^2_{L^2(X,\gamma)} + \|\ell\|^2_{L^2(X,\mu)}\right)\right\}.\end{aligned}$$

for every $\ell \in X^*$. Using the notation of Remark 1.1.14, for every $B \in \mathscr{B}(X)$ we have

$$\begin{aligned}\{(x,y) \in X \times X : h(x,y) \in B\}_x = B - x, \\ \{(x,y) \in X \times X : h(x,y) \in B\}^y = B - y.\end{aligned}$$

Applying the Fubini Theorem to the characteristic functions of $h^{-1}(B)$ we deduce that the convolution measure is given by (2.2.5).

To show (iii), set $\mu := (\gamma \otimes \gamma) \circ R_\theta^{-1}$; taking into account that for any $\ell \in (X \times X)^*$ we have

$$\begin{aligned}\ell(R_\theta(x,y)) =& \ell(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \\ =& \underbrace{\ell(x,0)\cos\theta - \ell(0,x)\sin\theta}_{=:f_\theta(x)} + \underbrace{\ell(y,0)\sin\theta + \ell(0,y)\cos\theta}_{=:g_\theta(y)}\end{aligned}$$

we find

$$\begin{aligned}\hat{\mu}(\ell) &= \int_{X \times X} \exp\{i\ell(R_\theta(x,y))\}(\gamma \otimes \gamma)(d(x,y)) \\ &= \int_X \exp\{if_\theta(x)\}\gamma(dx) \int_X \exp\{ig_\theta(y)\}\gamma(dy) \\ &= \exp\left\{-\frac{1}{2}\left(B_\gamma(f_\theta, f_\theta) + B_\gamma(g_\theta, g_\theta)\right)\right\}.\end{aligned}$$

since B_γ is bilinear,

$$B_\gamma(f_\theta, f_\theta) + B_\gamma(g_\theta, g_\theta) = B_\gamma(f_0, f_0) + B_\gamma(g_0, g_0),$$

where $f_0(x) = \ell(x,0)$, $g_0(y) = \ell(0,y)$ and

$$B_\gamma(f_0, f_0) + B_\gamma(g_0, g_0) = B_{\gamma\otimes\gamma}(\ell, \ell)$$

by the proof of statement (i).

To prove (iv), we notice that the laws of $\gamma \otimes \gamma$ under the projection maps $p_1, p_2 : X \times X \to X$,

$$p_1(x,y) := x, \qquad p_2(x,y) := y$$

are γ by definition of product measure. Since $\phi_i = p_i \circ R_\theta$, $i = 1, 2$ we deduce

$$(\gamma \otimes \gamma) \circ \phi_i^{-1} = (\gamma \otimes \gamma) \circ (p_i \circ R_\theta)^{-1} = ((\gamma \otimes \gamma) \circ R_\theta^{-1}) \circ p_i^{-1} = (\gamma \otimes \gamma) \circ p_i^{-1} = \gamma.$$

□

2.3 The Fernique Theorem

In this section we prove the Fernique Theorem; we start by proving it in the case of centred Gaussian measure and then we extend the result to any Gaussian measure.

Theorem 2.3.1 (Fernique)**.** *Let γ be a centred Gaussian measure on a separable Banach space X. Then there exists $\alpha > 0$ such that*

$$\int_X \exp\{\alpha \|x\|^2\}\, \gamma(dx) < \infty.$$

Proof. If γ is a Dirac measure the result is trivial, therefore we may assume that this is not the case. The idea of the proof is to show that the measures of suitable annuli decay fast enough to compensate the growth of the exponential function under the integral. Let us fix $t > \tau > 0$ and let us estimate $\gamma(\{\|x\| \le \tau\})\gamma(\{\|x\| > t\})$. Using property (iii) of Proposition 2.2.7 with $\theta = -\frac{\pi}{4}$, we obtain

$$\begin{aligned}
&\gamma(\{x \in X : \|x\| \le \tau\})\gamma(\{x \in X : \|x\| > t\}) \\
&= (\gamma \otimes \gamma)\left(\{(x, y) \in X \times X : \|x\| \le \tau\} \times \{(x, y) \in X \times X : \|y\| > t\}\right) \\
&= (\gamma \otimes \gamma)\left(\{\|x\| \le \tau\} \cap \{\|y\| > t\}\right) \\
&= (\gamma \otimes \gamma)\left(\Big\{\frac{\|x - y\|}{\sqrt{2}} \le \tau\Big\} \cap \Big\{\frac{\|x + y\|}{\sqrt{2}} > t\Big\}\right).
\end{aligned}$$

The triangle inequality yields $\|x\|, \|y\| \ge \frac{\|x+y\|}{2} - \frac{\|x-y\|}{2}$, which implies the inclusion

$$\Big\{\frac{\|x - y\|}{\sqrt{2}} \le \tau\Big\} \cap \Big\{\frac{\|x + y\|}{\sqrt{2}} > t\Big\} \subset \Big\{\|x\| > \frac{t - \tau}{\sqrt{2}}\Big\} \cap \Big\{\|y\| > \frac{t - \tau}{\sqrt{2}}\Big\}.$$

As a consequence, we have the estimate

$$\gamma(\{\|x\| \le \tau\})\gamma(\{\|x\| > t\}) \le \gamma\left(X \setminus \overline{B}\Big(0, \frac{t - \tau}{\sqrt{2}}\Big)\right)^2. \tag{2.3.1}$$

We leave as an exercise, Exercise 2.1, the fact that if γ is not a Dirac measure, then $\gamma(\overline{B}(0, \tau)) < 1$ for any $\tau > 0$. Let us fix $\tau > 0$ such that $c := \gamma\big(\overline{B}(0, \tau)\big) \in (1/2, 1)$ and set

$$\begin{aligned}
&\alpha := \frac{1}{24\tau^2} \log\Big(\frac{c}{1 - c}\Big), \\
&t_0 := \tau, \quad t_n := \tau + \sqrt{2} t_{n-1} = \tau(1 + \sqrt{2})\big(\sqrt{2}^{n+1} - 1\big), \qquad n \ge 1.
\end{aligned}$$

Applying estimate (2.3.1) with $t = t_n$ and recalling that $\frac{t_n - \tau}{\sqrt{2}} = t_{n-1}$, we obtain

$$\gamma(X \setminus \bar{B}(0, t_n)) \le \frac{\gamma(X \setminus \bar{B}(0, t_{n-1}))^2}{\gamma(\bar{B}(0, \tau))} = \Big(\frac{\gamma(X \setminus \bar{B}(0, t_{n-1}))}{c}\Big)^2 c$$

and iterating

$$\gamma(X \setminus \bar{B}(0, t_n)) \le c \left(\frac{1-c}{c}\right)^{2^n}.$$

Therefore

$$\begin{aligned}\int_X \exp\{\alpha\|x\|^2\}\,\gamma(dx) = &\int_{\overline{B}(0,\tau)} \exp\{\alpha\|x\|^2\}\,\gamma(dx) + \\ &+ \sum_{n=0}^{\infty} \int_{\overline{B}(0,t_{n+1})\setminus\overline{B}(0,t_n)} \exp\{\alpha\|x\|^2\}\,\gamma(dx) \\ \le & c\exp\{\alpha\tau^2\} + \sum_{n=0}^{\infty} \exp\{\alpha t_{n+1}^2\}\,\gamma(X \setminus \overline{B}(0, t_n)).\end{aligned}$$

Since $(\sqrt{2}^{n+2} - 1)^2 \le 2^{n+2}$ for every $n \in \mathbb{N}$,

$$\begin{aligned}\int_X \exp\{\alpha\|x\|^2\}\,\gamma(dx) &\le c\Big(\exp\{\alpha\tau^2\} + \sum_{n=0}^{\infty} \exp\{4\alpha\tau^2(1+\sqrt{2})^2 2^n\}\Big(\frac{1-c}{c}\Big)^{2^n}\Big) \\ &= c\Big(\exp\{\alpha\tau^2\} + \sum_{n=0}^{\infty} \exp\Big\{2^n\Big(\log\frac{1-c}{c} + 4\alpha\tau^2(1+\sqrt{2})^2\Big)\Big\}\Big) \\ &= c\Big(\exp\{\alpha\tau^2\} + \sum_{n=0}^{\infty} \exp\Big\{2^n\Big(\frac{1}{2} - \frac{\sqrt{2}}{3}\Big)\log\Big(\frac{1-c}{c}\Big)\Big\}\Big).\end{aligned}$$

The last series is convergent because $c > 1/2$ and hence $\log\left(\frac{1-c}{c}\right) < 0$. □

The validity of the Fernique Theorem can be extended to any Gaussian measure, not necessarily centred.

Corollary 2.3.2. *Let γ be a Gaussian measure on a separable Banach space X. Then there exists $\alpha > 0$ such that*

$$\int_X \exp\{\alpha\|x\|^2\}\,\gamma(dx) < +\infty.$$

Proof. Let us set $\mu(B) = \gamma(-B)$ for any $B \in \mathscr{B}(X)$. According to (2.2.5), the measure $\gamma_1 = \gamma * \mu$ is given by $(\gamma \otimes \mu) \circ h^{-1}$, $h(x, y) = x + y$, and is a centred Gaussian measure. Therefore, there exists $\alpha_1 > 0$ such that

$$\begin{aligned}+\infty > \int_X \exp\{\alpha_1\|x\|^2\}\,\gamma_1(dx) &= \int_{X\times X} \exp\{\alpha_1\|x+y\|^2\}\,(\gamma\otimes\mu)(d(x,y)) \\ = \int_{X\times X} \exp\{\alpha_1\|x-y\|^2\}\,(\gamma\otimes\gamma)(d(x,y)) &= \int_X \gamma(dy) \int_X \exp\{\alpha_1\|x-y\|^2\}\,\gamma(dx).\end{aligned}$$

Then, for γ–a.e. $y \in X$,

$$\int_X \exp\{\alpha_1 \|x-y\|^2\}\,\gamma(dx) < +\infty.$$

Using the inequality $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, which holds for any a, b and $\varepsilon > 0$, we get

$$\|x\|^2 \le \|x-y\|^2 + \|y\|^2 + 2\|x-y\|\|y\| \le (1+\varepsilon)\|x-y\|^2 + \Big(1+\frac{1}{\varepsilon}\Big)\|y\|^2.$$

For any $\alpha \in (0, \alpha_1)$, by setting $\varepsilon = \frac{\alpha_1}{\alpha} - 1$, we obtain

$$\int_X \exp\{\alpha\|x\|^2\}\,\gamma(dx) \le \exp\left\{\frac{\alpha\alpha_1}{\alpha_1-\alpha}\|y\|^2\right\}\int_X \exp\{\alpha_1\|x-y\|^2\}\,\gamma(dx).$$

Then for any $\alpha < \alpha_1$,

$$\int_X \exp\{\alpha\|x\|^2\}\,\gamma(dx) < +\infty.$$

□

As a first application of the Fernique theorem, we notice that for every $1 \le p < \infty$ we have

$$\int_X \|x\|^p\,\gamma(dx) < +\infty \tag{2.3.2}$$

since $\|x\|^p \le c_{\alpha,p}\exp\{\alpha\|x\|^2\}$ for all $x \in X$ and for some constant $c_{\alpha,p}$ depending on α and p only. We already know, through the definition of Gaussian measure, that the functions $f \in X^*$ belong to all $L^p(X,\gamma)$ spaces, for $1 \le p < \infty$. The Fernique Theorem tells us much more, since it gives a rather precise description of the allowed growth of the functions in $L^p(X,\gamma)$. Moreover, estimate (2.3.2) has important consequences on the functions a_γ and B_γ.

Proposition 2.3.3. *If γ is a Gaussian measure on a separable Banach space X, then $a_\gamma : X^* \to \mathbb{R}$ and $B_\gamma : X^* \times X^* \to \mathbb{R}$ are continuous. In addition, there exists $a \in X$ representing a_γ, i.e., such that*

$$a_\gamma(f) = f(a), \qquad \forall f \in X^*.$$

Proof. Let us define

$$c_1 := \int_X \|x\|\,\gamma(dx), \qquad c_2 := \int_X \|x\|^2\,\gamma(dx). \tag{2.3.3}$$

Then, if $f, g \in X^*$

$$\begin{aligned} |a_\gamma(f)| &\le \|f\|_{X^*}\int_X \|x\|\,\gamma(dx) = c_1\|f\|_{X^*}, \\ |B_\gamma(f,g)| &\le \int_X |f(x)-a_\gamma(f)||g(x)-a_\gamma(g)|\,\gamma(dx) \\ &\le \|f\|_{X^*}\|g\|_{X^*}\int_X (\|x\|+c_1)^2\,\gamma(dx) = (c_2+3c_1^2)\|f\|_{X^*}\|g\|_{X^*}. \end{aligned}$$

To show that a_γ can be represented by an element $a \in X$, by the general duality theory (see e.g. [Br, Proposition 3.14]), it is enough to show that the map $f \mapsto a_\gamma(f)$ is weakly* continuous on X^*, i.e., continuous with respect to the duality $\sigma(X^*, X)$. Moreover, since X is separable, by [Br, Theorem 3.28] weak* continuity is equivalent to continuity along weak* convergent sequences.

Let (f_j) be a sequence weakly* convergent to f, i.e.,

$$f_j(x) \to f(x), \qquad \forall x \in X.$$

By the Uniform Boundedness Principle,

$$\sup_{j\in\mathbb{N}} \|f_j\|_{X^*} < +\infty,$$

and by the Lebesgue Dominated Convergence Theorem, we deduce

$$\lim_{j\to+\infty} a_\gamma(f_j) = \lim_{j\to+\infty} \int_X f_j(x)\,\gamma(dx) = \int_X f(x)\,\gamma(dx) = a_\gamma(f).$$

□

The space X^* is continuously embedded in $L^2(X, \gamma)$ by the map $j : X^* \to L^2(X, \gamma)$,

$$j(f) = f - a_\gamma(f), \qquad f \in X^*.$$

The operator j is continuous because $\|j(f)\|_{L^2(X,\gamma)} \le (1 + c_1)\|f\|_{X^*}$, where c_1 is defined in (2.3.3). We define the *reproducing kernel* as the closure of the range of this embedding,

Definition 2.3.4 (Reproducing kernel)**.** *The* reproducing kernel *is defined by*

$$X^*_\gamma := \text{ the closure of } j(X^*) \text{ in } L^2(X, \gamma), \tag{2.3.4}$$

i.e. X^*_γ *consists of all limits in* $L^2(X, \gamma)$ *of sequences of functions* $j(f_h) = f_h - a_\gamma(f_h)$ *with* $(f_h) \subset X^*$.

The definitions of characteristic function, mean and covariance can be extended to the reproducing kernel.

Proposition 2.3.5. *Let* γ *be a Gaussian measure on a separable Banach space. Then, the functions* $\widehat{\gamma}$, a_γ *and* B_γ *admit extensions defined on* X^*_γ *that are continuous with respect to the* $L^2(X, \gamma)$ *topology. In particular,* $a_\gamma = 0$ *on* X^*_γ *and*

$$\widehat{\gamma}(f) = \exp\Big\{-\frac{1}{2}\|f\|^2_{L^2(X,\gamma)}\Big\}, \qquad \forall f \in X^*_\gamma.$$

Proof. The extensions of a_γ and B_γ are obvious since the maps

$$f \mapsto a_\gamma(f) = \int_X f(x)\,\gamma(dx),$$

and
$$(f,g) \mapsto B_\gamma(f,g) = \int_X \Big[f(x) - a_\gamma(f)\Big]\,\Big[g(x) - a_\gamma(g)\Big]\,\gamma(dx)$$
are continuous in the $L^2(X,\gamma)$ topology. So, for $f,g \in X^*_\gamma$ we have $a_\gamma(f) = 0$,
$$B_\gamma(f,g) = \int_X f(x)g(x)\gamma(dx) = \langle f,g\rangle_{L^2(X,\gamma)},$$
and
$$\hat\gamma(f) = \int_X \exp\{if(x)\}\,\gamma(dx).$$
Let $g_h = j(f_h)$, $f_h \in X^*$, be a sequence of functions converging to f in $L^2(X,\gamma)$. Then, $\lim_{h\to\infty} a_\gamma(g_h) = 0$. Using the fact that the map $t \mapsto e^{it}$ is 1-Lipschitz, we have
$$\begin{aligned}\Big|\int_X \exp\{ig_h(x)\} - \exp\{if(x)\}\,\gamma(dx)\Big| &\le \int_X |\exp\{ig_h(x)\} - \exp\{if(x)\}|\,\gamma(dx)\\ &\le \int_X |g_h(x) - f(x)|\,\gamma(dx)\\ &\le \Big(\int_X |g_h(x) - f(x)|^2\,\gamma(dx)\Big)^{1/2} \to 0\end{aligned}$$
Therefore,
$$\hat\gamma(f) = \lim_{h\to\infty}\hat\gamma(g_h) = \lim_{h\to\infty}\exp\Big\{a_\gamma(g_h) - \frac12 B_\gamma(g_h,g_h)\Big\} = \exp\Big\{-\frac12\|f\|_{L^2(X,\gamma)}\Big\}.$$
□

Notice that if γ is nondegenerate then two different elements of X^* define two different elements of X^*_γ, but if γ is degenerate two different elements of X^* may define elements coinciding γ-a.e.

By Proposition 2.3.5, we may define the operator $R_\gamma : X^*_\gamma \to (X^*)'$ by
$$R_\gamma f(g) := \int_X f(x)[g(x) - a_\gamma(g)]\,\gamma(dx), \qquad f \in X^*_\gamma,\ g \in X^*. \tag{2.3.5}$$
Observe that
$$R_\gamma f(g) = \langle f, g - a_\gamma(g)\rangle_{L^2(X,\gamma)}. \tag{2.3.6}$$
It is important to notice that indeed R_γ maps X^*_γ into X.

Proposition 2.3.6. *The range of R_γ is contained in X, i.e., for every $f \in X^*_\gamma$ there is $y \in X$ such that $R_\gamma f(g) = g(y)$ for all $g \in X^*$.*

Proof. As in the proof of Proposition 2.3.3, we show that for every $f \in X^*_\gamma$ the map $g \mapsto R_\gamma f(g)$ is weakly* continuous on X^*, i.e., continuous with respect to the duality $\sigma(X^*,X)$. By the general duality theory (see e.g. [Br, Proposition 3.14]) we deduce that $R_\gamma f \in X$. Recall that, since X is separable, weak* continuity is equivalent to continuity along weak* convergent sequences. Let then $(g_k) \subset X^*$ be weakly* convergent to g, i.e., $g_k(x) \to g(x)$ for every $x \in X$. Then, by the Uniform Boundedness Principle the sequence (g_k) is bounded in X^* and by the Dominated Convergence Theorem $a_\gamma(g_k) \to a_\gamma(g)$ and

$$R_\gamma f(g_k) = \int_X f(x)[g_k(x) - a_\gamma(g_k)]\,\gamma(dx) \quad \longrightarrow \quad \int_X f(x)[g(x) - a_\gamma(g)]\,\gamma(dx) = R_\gamma f(g).$$

□

Remark 2.3.7. Thanks to Proposition 2.3.6, we can identify $R_\gamma f$ with the element $y \in X$ representing it, i.e. we shall write

$$R_\gamma f(g) = g(R_\gamma f), \qquad \forall g \in X^*.$$

2.4 Exercises

Exercise 2.1. Prove that if γ is a Gaussian measure and γ is not a Dirac measure, then for any $r > 0$ and $x \in X$,

$$\gamma(B(x,r)) < 1.$$

Exercise 2.2. Let X be an infinite dimensional Banach space. Prove that there is no nontrivial measure μ on X invariant under translations and such that $\mu(B) > 0$ for any ball B. *Hint:* modify the construction described in the Hilbert case using a sequence of elements in the unit ball having mutual distance 1/2.

Exercise 2.3. Prove that a Gaussian measure on a Banach space is degenerate iff there exists $X^* \ni f \neq 0$ such that $\hat{\gamma}(f) = 1$ and hence iff there exists a proper closed subspace $V \subset X$ with $\gamma(V) = 1$.

Exercise 2.4. Let γ be centred. Prove that for any choice $f_1, \dots, f_d \in X^*$, setting

$$P(x) = (f_1(x), \dots, f_d(x)),$$

$\gamma \circ P^{-1}$ is the Gaussian measure $\mathscr{N}(0, Q)$, with $Q_{i,j} = \langle f_i, f_j \rangle_{L^2(X,\gamma)}$. If $L : \mathbb{R}^d \to \mathbb{R}^n$ is another linear map, compute the covariance matrix of $\gamma \circ (L \circ P)^{-1}$.

Exercise 2.5. Let $\gamma = \mathscr{N}(a, Q)$ be a nondegenerate Gaussian probability measure on $\mathbb{R}^d$. Show that

$$\int_{\mathbb{R}^d} |x|^4\,\gamma(dx) = (\mathrm{Tr}\,Q)^2 + 2\mathrm{Tr}(Q^2).$$

Hint: Consider the function $F(\varepsilon) = \int_{\mathbb{R}^d} e^{\varepsilon |x|^2} \gamma(dx)$ and compute $F''(0)$.

Exercise 2.6. Let γ be a centred Gaussian measure on a separable Banach space X. Compute the integrals

$$\int_X e^{f(x)}\,\gamma(dx), \qquad \int_X \big(f(x)\big)^k\,\gamma(dx),\ k \in \mathbb{N}$$

for every $f \in X^*$.

Bibliography

[Br] H. Brezis: *Functional Analysis, Sobolev spaces and partial differential equations,* Springer, 2011.

[D] R. M. Dudley: *Real Analysis and Probabillity,* Cambridge University Press, 2004.

[DS1] N. Dunford, J. T. Schwartz: *Linear operators I*, Wiley, 1958.