Solutions of the Exercises of Lecture 14

Team of the TU Dresden

Exercise 14.1

We show in a more general context: If $(T(t))_{t\geq 0}$ is a C_0-semigroup on a Banach space X, with generator L, and $\lambda \in \mathbb{R}$ is an eigenvalue of L with eigenvector $x \in X$, then

$$T(t)x = e^{\lambda t}x \qquad (t \geq 0). \tag{1}$$

Indeed, the function $u(t) := e^{\lambda t}x$ satisfies $u'(t) = \lambda e^{\lambda t}x = Lu(t)$ $(t \geq 0)$, $u(0) = x$, and therefore Lemma 11.1.7 implies (1).
The application of this result yields (14.1.4): $T_2(t)H_\alpha = e^{-|\alpha|t}H_\alpha$ $(t \geq 0)$. This holds because $L_2H_\alpha = -|\alpha|H_\alpha$ $(\alpha \in \Lambda)$.

Exercise 14.2.

Show that $W^{1,p}(\mathbb{R},\gamma_1)$ is not contained in $L^{p+\epsilon}(\mathbb{R},\gamma_1)$ for every $p \in [1,\infty)$ and $\epsilon > 0$.

Proposition 1. *Let $1 \le p, q < \infty$, and $f_q\colon \mathbb{R} \to \mathbb{R}, x \mapsto \mathrm{e}^{x^2/(2q)}$. Then $f_q \in L^p(\mathbb{R},\gamma_1)$ if and only if $p < q$, and $f_q \in W^{1,p}(\mathbb{R},\gamma_1)$ if and only if $p < q$.*

Proof. From

$$|f_q(x)|^p \mathrm{e}^{-\frac{x^2}{2}} = \mathrm{e}^{-\frac{q-p}{q}\frac{x^2}{2}} \qquad (x \in \mathbb{R})$$

we obtain $f_q \in L^p(\mathbb{R},\gamma_1)$ if and only if $p < q$. Further

$$|f_q'(x)|^p \mathrm{e}^{-\frac{x^2}{2}} = (|x|/q)^p \mathrm{e}^{-\frac{q-p}{q}\frac{x^2}{2}} \qquad (x \in \mathbb{R})$$

holds yielding $f_q' \in L^p(\mathbb{R},\gamma_1)$ if and only if $p < q$. Together with the previous observation we conclude $f_q \in W^{1,p}(\mathbb{R},\gamma_1)$ if and only if $p < q$. □

Taking $1 < q < \infty$, we get finally that f_q belongs to $W^{1,p}(\mathbb{R},\gamma_1)$ for all $1 \le p < q$ but not to $L^q(\mathbb{R},\gamma_1)$, which already yields a solution of the exercise. We now show additionally that, for $1 \le p < \infty$ we can also find a function in $W^{1,p}(\mathbb{R},\gamma_1)$ which does not belong to $L^q(\mathbb{R},\gamma_1)$ for any $q > p$.

Proposition 2. *Let $1 \le p, q < \infty$, and $g_p\colon \mathbb{R} \to \mathbb{R}, x \mapsto \mathrm{e}^{x^2/(2p)}/(1+x^4)$. Then $g_p \in L^q(\mathbb{R},\gamma_1)$ if and only if $q \le p$, and $g_p \in W^{1,q}(\mathbb{R},\gamma_1)$ if and only if $q \le p$.*

Proof. As in the previous proof, we get $g_p \in L^q(\mathbb{R},\gamma_1)$ if and only if $q \le p$. Since

$$g_p'(x) = \mathrm{e}^{\frac{x^2}{2p}}\left(\frac{x}{p(1+x^4)} - \frac{4x^3}{(1+x^4)^2}\right) = \frac{x^5 - 4px^3 + x}{p(1+x^4)^2}\mathrm{e}^{\frac{x^2}{2p}} \qquad (x \in \mathbb{R})$$

and the mapping $\mathbb{R} \ni x \mapsto (x^5 - 4px^3 + x)/(1+x^4)^2 \in \mathbb{R}$ is $L^r(\mathbb{R})$-integrable for every $r \ge 1$, we obtain $g_p' \in L^q(\mathbb{R},\gamma_1)$ and consequently $g_p \in W^{1,q}(\mathbb{R},\gamma_1)$ if and only if $q \le p$. □

Incidentally, the function $h_p\colon \mathbb{R} \to \mathbb{R}, x \mapsto \mathrm{e}^{x^2/2p}/(1+x^2)$ also gives a positive answer in the case $1 < p < \infty$. Unfortunately, h_1 does not belong to $W^{1,1}(\mathbb{R},\gamma_1)$ because $\mathbb{R} \ni x \mapsto x/(1+x^2) \in \mathbb{R}$ is not integrable.

Exercise 14.3

Theorem. *Let X be a separable Banach space equipped with a nondegenerate centred Gaussian measure γ and let $p \in [1, \infty)$. Let $f \in W^{1,p}(X, \gamma)$. For $k \in \mathbb{N}$ define $\varphi_k(r) := \sqrt{r^2 + 1/k}$ ($r \in \mathbb{R}$). Then $(\varphi_k \circ f)_{k \in \mathbb{N}}$ converges to $|f|$ in $W^{1,p}(X, \gamma)$.*

Proof. At first we show L^p-convergence of the sequence $(\varphi_k \circ f)_{k\in\mathbb{N}}$ and then the convergence of the H-gradients.

Note that φ_k is in $C^1(\mathbb{R})$ for every $k \in \mathbb{N}$. For every $x \in X$ we have

$$\lim_{k\to\infty} \big||f(x)| - (\varphi_k \circ f)(x)\big| = \lim_{k\to\infty} \left||f(x)| - \sqrt{f(x)^2 + 1/k}\right| = 0$$

From

$$0 \leq \varphi_k \circ f = \sqrt{f^2 + 1/k} \leq |f| + 1 \in L^p(X, \gamma)$$

it follows that $\varphi_k \circ f \in L^p(X, \gamma)$ for all $k \in \mathbb{N}$, and the dominated convergence theorem implies $\varphi_k \circ f \to |f|$ in $L^p(X, \gamma)$ ($k \to \infty$).

By Definition 9.3.9 there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}C_b^1(X)$ such that $f_n \xrightarrow{n} f$ in L^p and $\nabla_H f_n \xrightarrow{n} \nabla_H f$ in $L^p(X, \gamma; H)$. There exists a subsequence of (f_n) (again denoted by (f_n)) which converges to f almost everywhere and whose gradients $\nabla_H f_n$ converge to $\nabla_H f$ almost everywhere. Additionally the subsequence (f_n) can be chosen to satisfy $|\nabla_H f_n|_H \leq F$ with $F \in L^p(X, \gamma)$.

For these f_n we can write the H-Gradient of $\varphi_k \circ f_n$ as

$$\nabla_H(\varphi_k \circ f_n) = (\varphi_k' \circ f_n)\nabla f_n.$$

Since (f_n) and $(\nabla_H f_n)$ converge to f and $\nabla_H f$, respectively, almost everywhere, and since φ_k' is continuous and bounded it follows that

$$(\varphi_k' \circ f_n)\nabla_H f_n \xrightarrow{n} (\varphi_k' \circ f)\nabla_H f$$

almost everywhere and by dominated convergence also in $L^p(X, \gamma;$ H).

Thus we have $\nabla_H(\varphi_k \circ f) = (\varphi_k' \circ f)\nabla_H f$ for all $f \in W^{1,p}(X, \gamma)$ and

$$\nabla_H(\varphi_k \circ f) = \nabla_H(\sqrt{f^2 + 1/k}) \xrightarrow{k\to\infty} \operatorname{sgn}(f)\nabla_H f.$$

Summarising, we have shown that $\varphi_k \circ f \to |f|$ in $W^{1,p}(X, \gamma)$ as $k \to \infty$ (and that $\nabla_H|f| = \operatorname{sgn}(f)\nabla_H f$). $\square$

Exercise 14.4

Let $p > 1$. We first take $f \in \Sigma$ (recall Σ from the second paragraph in section 13.2) and for $\varepsilon > 0$ define $g_\varepsilon := f(f^2+\varepsilon)^{\frac{p}{2}-1}$. Then $g_\varepsilon \in C_b^1(X)$ by the definition of Σ and formula (13.2.1). Hence, we can apply formula (13.2.5) to f and g_ε and get

$$-\int_X g_\varepsilon L_p f \,\mathrm{d}\gamma = -\int_X g_\varepsilon L_2 f \,\mathrm{d}\gamma = \int_X \big[\nabla_H f, \nabla_H g_\varepsilon\big]_H \,\mathrm{d}\gamma,$$

since $L_p f(x) = \operatorname{div}_\gamma \nabla_H f(x) = L_2 f(x)$ for γ-a. e. $x \in X$ by Proposition 13.2.1. Inserting the definition of g_ε yields

$$-\int_X f(f^2+\varepsilon)^{\frac{p}{2}-1} L_p f \,\mathrm{d}\gamma = \int_X \Big((f^2+\varepsilon)^{\frac{p}{2}-1} + (p-2)f^2(f^2+\varepsilon)^{\frac{p}{2}-2}\Big)|\nabla_H f|_H^2 \,\mathrm{d}\gamma. \quad (*)$$

Now, let $f \in D(L_p)$. Since, by Theorem 13.2.2, Σ is a core of L_p, there is a sequence (f_n) in $L^p(X,\gamma)$, such that $f_n \to f$ and $L_p f_n \to L_p f$ in $L^p(X,\gamma)$. Without a proof we use, that we have $\nabla_H f_n \to \nabla_H f$ in $L^p(X,\gamma)$, too. (Indeed, it was mentioned at the end of Lecture 13 that, for $1 < p < \infty$, one has $D(L_p) = W^{2,p}(X,\gamma)$, and this implies the previous statement.) Then (after extracting a subsequence) we can find $F \in L^p(X,\gamma)$, such that

$$|f_n| + 1 \le F, \qquad |L_p f_n| \le F \qquad \text{and} \qquad |\nabla_H f_n| \le F.$$

Because formula $(*)$ holds for every f_n, by the Dominated Convergence Theorem it holds for f (and $\varepsilon < 1$), too. For the right hand side of $(*)$, however, this reasoning only works for $p \ge 2$! If $1 < p < 2$ (since by the calculation below all the integrands are positive), Fatou's Lemma yields the inequality

$$-\int_X f(f^2+\varepsilon)^{\frac{p}{2}-1} L_p f \,\mathrm{d}\gamma \ge \int_X \Big((f^2+\varepsilon)^{\frac{p}{2}-1} + (p-2)f^2(f^2+\varepsilon)^{\frac{p}{2}-2}\Big)|\nabla_H f|_H^2 \,\mathrm{d}\gamma. \quad (**)$$

On the left hand side of formula $(*)$ (or $(**)$) the Dominated Convergence Theorem yields

$$\int_X f(f^2+\varepsilon)^{\frac{p}{2}-1} L_p f \,\mathrm{d}\gamma \xrightarrow{\varepsilon\to 0} \int_X f|f|^{p-2} L_p f \,\mathrm{d}\gamma. \quad (***)$$

Looking at the right hand side of formula $(*)$ (or $(**)$) we see

$$\begin{aligned}
&\int_X \Big((f^2+\varepsilon)^{\frac{p}{2}-1} + (p-2)f^2(f^2+\varepsilon)^{\frac{p}{2}-2}\Big)|\nabla_H f|_H^2 \,\mathrm{d}\gamma \\
&\quad = \int_X \Big((f^2+\varepsilon)^{\frac{p}{2}-1} + (p-2)f^2(f^2+\varepsilon)^{\frac{p}{2}-2}\Big)|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}} \,\mathrm{d}\gamma \\
&\qquad + \varepsilon^{\frac{p}{2}-1} \int_X |\nabla_H f|_H^2 \mathbb{1}_{\{f=0\}} \,\mathrm{d}\gamma.
\end{aligned}$$

But $\int_X |\nabla_H f|_H^2 \mathbb{1}_{\{f=0\}} \,\mathrm{d}\gamma = 0$, for $\nabla_H f(x) = 0$ γ-a. e. on $\{f = 0\}$ by Exercise 10.3. Thus we have

$$\begin{aligned}
&\int_X \Big((f^2+\varepsilon)^{\frac{p}{2}-1} + (p-2)f^2(f^2+\varepsilon)^{\frac{p}{2}-2}\Big)|\nabla_H f|_H^2 \,\mathrm{d}\gamma \\
&\qquad = \int_X \Big((f^2+\varepsilon)^{\frac{p}{2}-1} + (p-2)f^2(f^2+\varepsilon)^{\frac{p}{2}-2}\Big)|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}} \,\mathrm{d}\gamma \\
&\qquad = \int_X \Big(1 + (p-2)\frac{f^2}{f^2+\varepsilon}\Big)(f^2+\varepsilon)^{\frac{p}{2}-1}|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}} \,\mathrm{d}\gamma.
\end{aligned}$$

To see that this expression converges in the right way, we use the following consequence of the Monotone Convergence Theorem and the Dominated Convergence Theorem.

Lemma. *Let (f_n), (φ_n) sequences of positive, measurable fuctions such that*

- *(f_n) is monotone,*
- *there are constants $0 < \alpha < \beta < \infty$, such that $\alpha \le \varphi_n \le \beta$ for all $n \in \mathbb{N}$,*
- *(φ_n) converges pointwise to a measurable function φ,*
- *there is a constant $c > 0$, such that $\int_X \varphi_n f_n \,\mathrm{d}\gamma \le c$ for all $n \in \mathbb{N}$.*

Let f be the pointwise limit of (f_n). Then φf is integrable and $\int_X \varphi_n f_n \,\mathrm{d}\gamma \to \int_X \varphi f \,\mathrm{d}\gamma$.

In our case $\varepsilon \mapsto (f^2+\varepsilon)^{\frac{p}{2}-1}|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}}$ is monotone and

$$0 < \begin{cases} 1 \\ p-1 \end{cases} \le 1 + (p-2)\frac{f^2}{f^2+\varepsilon} \le \begin{cases} p-1, & p \ge 2, \\ 1, & p < 2. \end{cases}$$

Since the left hand side of formula $(*)$ converges as ε tends to 0, the right hand side stays bounded and the Lemma above gives

$$\begin{aligned}
&\int_X \Big((f^2+\varepsilon)^{\frac{p}{2}-1} + (p-2)f^2(f^2+\varepsilon)^{\frac{p}{2}-2}\Big)|\nabla_H f|_H^2 \,\mathrm{d}\gamma \\
&\qquad = \int_X \Big(1 + (p-2)\frac{f^2}{f^2+\varepsilon}\Big)(f^2+\varepsilon)^{\frac{p}{2}-1}|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}} \,\mathrm{d}\gamma \\
&\qquad \xrightarrow{\varepsilon\downarrow 0} (p-1)\int_X |f|^{p-2}|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}} \,\mathrm{d}\gamma.
\end{aligned}$$

Together with formula $(*)$ (or $(**)$) and formula $(***)$ we conclude

$$-\int_X f|f|^{p-2}L_p f \,\mathrm{d}\gamma = (p-1)\int_X |f|^{p-2}|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}} \,\mathrm{d}\gamma$$

for $p \ge 2$ and

$$-\int_X f|f|^{p-2}L_p f \,\mathrm{d}\gamma \ge (p-1)\int_X |f|^{p-2}|\nabla_H f|_H^2 \mathbb{1}_{\{f\neq 0\}} \,\mathrm{d}\gamma$$

for $1 < p < 2$.

Comments.

1. We are able to show equality in formula (14.2.8) only in case $p \geq 2$. In case $1 < p < 2$ we get an inequality.

2. To proof formula (14.2.9), which is the aim of formula (14.2.8), this inequality suffices (see page 177, Lecture 14).

Note after completion of the solution. Diego Pallara communicated upon our request in the internet forum:

Equality (14.2.8) has a nice story. Even in the case of the Laplacean, it SEEMS obvious, but it is not, in the 1<p<2 case... In all the cases I know, as you say, the "≥" inequality is sufficient and therefore the "=" did not receive great interest. Indeed, it is true, and this has been proved by Giorgio Metafune and Chiara Spina in finite dimensions. (“An integration by parts formula in Sobolev spaces", Mediterr. j. math. **5**, 357–369 (2008)). Their proof works equally well in the Wiener case passing through cylindrical approximation, we shall quote the M-S paper in our revision.

Exercise 14.5

Let $f \in C_b^1(X)$ with $\int_X f \,\mathrm{d}\gamma = 0$. We choose $\varepsilon_0 > 0$ such that $-1/2 \leq \varepsilon_0 f \leq 1/2$, and in the following considerations we let $0 < \varepsilon \leq \varepsilon_0$, $f_\varepsilon := 1 + \varepsilon f$. Applying the Log-Sobolev-Inequality (14.2.7) to f_ε, we get

$$\int_X (1+\varepsilon f)^2 \log(1+\varepsilon f)\,\mathrm{d}\gamma \leq \|1+\varepsilon f\|_2^2 \log \|1+\varepsilon f\|_2 + \int_X |\nabla_H(1+\varepsilon f)|_H^2 \,\mathrm{d}\gamma.$$

Since $|\nabla_H(1+\varepsilon f)|_H^2 = \varepsilon^2 |\nabla_H f|_H^2$, we obtain the estimate

$$\frac{1}{\varepsilon^2}\int_X (1+\varepsilon f)^2 \log(1+\varepsilon f)\,\mathrm{d}\gamma \leq \frac{1}{\varepsilon^2}\|1+\varepsilon f\|_2^2 \log \|1+\varepsilon f\|_2 + \int_X |\nabla_H f|_H^2 \,\mathrm{d}\gamma. \qquad (*)$$

Concerning the first summand on the right hand side, we compute, using that $\langle f, 1\rangle = 0$,

$$\frac{1}{\varepsilon^2}\|1+\varepsilon f\|_2^2 \log \|1+\varepsilon f\|_2 = \frac{1}{2}\|1+\varepsilon f\|_2^2 \frac{1}{\varepsilon^2}\log(1+\varepsilon^2\|f\|_2^2) \to \frac{1}{2}\|f\|_2^2 \quad (\varepsilon \to 0).$$

For the left hand side of $(*)$ we have

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_X (1+\varepsilon f)^2 \log(1+\varepsilon f)\,\mathrm{d}\gamma \\
&= \frac{1}{\varepsilon^2}\int_X \log(1+\varepsilon f)\,\mathrm{d}\gamma + \frac{2}{\varepsilon}\int_X f\log(1+\varepsilon f)\;\mathrm{d}\gamma + \int_X f^2 \log(1+\varepsilon f)\,\mathrm{d}\gamma.
\end{aligned}$$

By dominated convergence, the last term converges to 0 and the next to last term converges to $2\|f\|_2^2$, as $\varepsilon \to 0$. For the first term, we use the Taylor formula for the logarithm,

$$\log(1+r) = r - \frac{r^2}{2} + \varphi(r) \qquad (-1/2 \leq r \leq 1/2),$$

with $|\varphi(r)| \leq c|r|^3$, for all $r \in (-1/2, 1/2)$ and some $c > 0$. This implies that

$$\frac{1}{\varepsilon^2}\int_X \log(1+\varepsilon f)\,\mathrm{d}\gamma = \frac{1}{\varepsilon}\int_X f\,\mathrm{d}\gamma - \frac{1}{2}\int_X f^2\,\mathrm{d}\gamma + \frac{1}{\varepsilon^2}\int_X \varphi(\varepsilon f)\,\mathrm{d}\gamma \to -\frac{1}{2}\|f\|_2^2,$$

because $\int_X f\,\mathrm{d}\gamma = 0$ and $\left|\varepsilon^{-2}\int_X \varphi(\varepsilon f)\,\mathrm{d}\gamma\right| \leq c\varepsilon \int_X |f|^3\,\mathrm{d}\gamma \to 0$, as $\varepsilon \to 0$.

Summarising, we end up with

$$-\frac{1}{2}\|f\|_2^2 + 2\|f\|_2^2 \leq \frac{1}{2}\|f\|^2 + \int_X |\nabla_H f|_H^2\,\mathrm{d}\gamma,$$

which implies the Poincaré inequality (14.2.11),

$$\int_X (f - \bar{f})^2\,\mathrm{d}\gamma \leq \int_X |\nabla_H f|_H^2\,\mathrm{d}\gamma.$$

Exercise 14.6
We prove the following Proposition.

Proposition. *Let $k \in \mathbb{N}_{>0}$ and $c > 0$ such that for each $f \in W^{1,2^k}(X,\gamma)$*

$$\int_X |f - \overline{f}|^{2^k} \, \mathrm{d}\gamma \leq c \int_X |\nabla_H f|^{2^k} \, \mathrm{d}\gamma.$$

Then, for each $p \in [2^k, 2^{k+1}]$ there exists $c_p > 0$ such that for every $f \in W^{1,p}(X,\gamma)$

$$\int_X |f - \overline{f}|^p \, \mathrm{d}\gamma \leq c_p \int_X |\nabla_H f|^p \, \mathrm{d}\gamma.$$

We remark that this proposition implies the Poincaré inequality for each $p > 2$ by induction and the fact, that it was already shown for $p = 2$ in Theorem 14.2.5. Now we come to the proof.

Proof. Let $p \in [2^k, 2^{k+1}]$ and $f \in W^{1,p}(X,\gamma)$. We have $\nabla_H |f|^{\frac{p}{2}} = \frac{p}{2}|f|^{\frac{p}{2}-1}\nabla_H f$ and thus, $|f|^{\frac{p}{2}} \in W^{1,2}(X,\gamma)$, since $\nabla_H f \in L^p(X,\gamma)$ and $|f|^{\frac{p}{2}-1} \in L^{\frac{2p}{p-2}}(X,\gamma)$ and thus, $\nabla_H |f|^{\frac{p}{2}} \in L^2(X,\gamma)$ follows by Hölders inequality. We apply Theorem 14.2.5 to $|f|^{\frac{p}{2}}$ and obtain

$$\int_X \left(|f|^{\frac{p}{2}} - \int_X |f|^{\frac{p}{2}} \, \mathrm{d}\gamma \right)^2 \mathrm{d}\gamma \leq \int_X \left| \nabla_H |f|^{\frac{p}{2}} \right|^2 \mathrm{d}\gamma.$$

Hence,

$$\|f\|_p^p = \int_X |f|^p \, \mathrm{d}\gamma \leq \left(\int_X |f|^{\frac{p}{2}} \, \mathrm{d}\gamma \right)^2 + \int_X \left| \nabla_H |f|^{\frac{p}{2}} \right|^2 \mathrm{d}\gamma, \tag{1}$$

where we have used

$$\int_X \left(|f|^{\frac{p}{2}} - \int_X |f|^{\frac{p}{2}} \, \mathrm{d}\gamma \right)^2 \mathrm{d}\gamma = \int_X |f|^p \, \mathrm{d}\gamma - \left(\int_X |f|^{\frac{p}{2}} \, \mathrm{d}\gamma \right)^2.$$

Since $\frac{p}{2} \leq 2^k$ and thus, $L^{2^k}(X,\gamma) \hookrightarrow L^{\frac{p}{2}}(X,\gamma)$ contractive, we can estimate the first term on the right hand side of (1) by

$$\left(\int_X |f|^{\frac{p}{2}} \, \mathrm{d}\gamma \right)^2 = \|f\|_{\frac{p}{2}}^p \leq \|f\|_{2^k}^p.$$

Moreover, we estimate the second term on the right hand side of (1) by using Hölders inequality as follows:

$$\begin{aligned}\int_X \left| \nabla_H |f|^{\frac{p}{2}} \right|^2 \mathrm{d}\gamma &= \frac{p^2}{4} \int_X |f|^{p-2} |\nabla_H f|^2 \, \mathrm{d}\gamma \\ &\leq \frac{p^2}{4} \|f\|_p^{p-2} \|\nabla_H f\|_p^2.\end{aligned}$$

Using now the inequality $a^{p-2}b^2 \leq a^p + b^p$ for each $a, b > 0$ (note that $p \geq 2$), we can estimate the latter product by

$$\begin{aligned}\frac{p^2}{4}\|f\|_p^{p-2}\|\nabla_H f\|_p^2 &= \frac{p^2}{4}\|\varepsilon^{\frac{1}{p}} f\|_p^{p-2}\|\varepsilon^{\frac{2-p}{2p}}\nabla_H f\|_p^2 \\ &\leq \frac{p^2}{4}\left(\varepsilon\|f\|_p^p + \varepsilon^{1-\frac{p}{2}}\|\nabla_H f\|_p^p\right)\end{aligned}$$

for each $\varepsilon > 0$. Summarizing, by choosing $\varepsilon = \frac{2}{p^2}$, we end up with the estimate

$$\int\limits_X |f|^p \,\mathrm{d}\gamma \leq \|f\|_{2^k}^p + \frac{1}{2}\|f\|_p^p + K_p\|\nabla_H f\|_p^p,$$

where $K_p \coloneqq \frac{1}{2}\left(\frac{2}{p^2}\right)^{-\frac{p}{2}}$. Thus, we infer

$$\|f\|_p^p \leq C_p\left(\|f\|_{2^k}^p + \|\nabla_H f\|_p^p\right)$$

with $C_p \coloneqq 2\max\{1, K_p\} = 2K_p$. Applying the latter inequality to the function $f - \overline{f}$ and using the assumption of the proposition, we derive

$$\begin{aligned}\|f - \overline{f}\|_p^p &\leq C_p\left(\|f - \overline{f}\|_{2^k}^p + \|\nabla_H f\|_p^p\right) \\ &\leq C_p\left(c^{\frac{p}{2^k}}\|\nabla_H f\|_{2^k}^p + \|\nabla_H f\|_p^p\right) \\ &\leq C_p(1 + c^{\frac{p}{2^k}})\|\nabla_H f\|_p^p,\end{aligned}$$

where we have used $L^p(X, \gamma) \hookrightarrow L^{2^k}(X, \gamma)$ contractive, since $2^k \leq p$. □