Lecture 14

More on Ornstein-Uhlenbeck operator and semigroup

In this lecture we go on in the study of the realisation of the Ornstein-Uhlenbeck operator and of the Ornstein-Uhlenbeck semigroup in L^p spaces. As in the last lectures, X is a separable Banach space endowed with a centred nondegenerate Gaussian measure γ, and H is the Cameron-Martin space. We use the notation of Lectures 12 and 13.

We start with the description of the spectrum of L_2. Although the domain of L_2 is not compactly embedded in $L^2(X, \gamma)$ if X is infinite dimensional, the spectrum of L_2 consists of a sequence of eigenvalues, and the corresponding eigenfunctions are the Hermite polynomials that we already encountered in Lecture 8. So, $L^2(X, \gamma)$ has an orthonormal basis made by eigenfunctions of L_2. This is used to obtain another representation formula for $T_2(t)$ and another characterisation of $D(L_2)$ in terms of Hermite polynomials.

In the second part of the lecture we present two important inequalities, the the Logarithmic Sobolev and Poincaré inequalities, that hold for C^1_b functions and are easily extended to Sobolev functions. They are used to prove summability improving properties and asymptotic behavior results for $T_p(t)$.

14.1 Spectral properties of L_2

Let $\{h_j : \ j \in \mathbb{N}\}$ be any orthonormal basis of H contained in $R_\gamma(X^*)$. We recall the definition of the Hermite polynomials, given in Lecture 8.

Λ is the set of of multi-indices $\alpha \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$, $\alpha = (\alpha_j)$, with finite length $|\alpha| = \sum_{j=1}^{\infty} \alpha_j < \infty$. For every $\alpha \in \Lambda$, $\alpha = (\alpha_j)$, the Hermite polynomial H_α is defined by

$$H_\alpha(x) = \prod_{j=1}^{\infty} H_{\alpha_j}(\hat{h}_j(x)), \quad x \in X.$$

where the polynomial H_{α_j} is defined in (8.1.1). By Lemma 8.1.2, for every $k \in \mathbb{N}$ we have

$$H_k''(\xi) - xH_k'(\xi) = -kH_k(\xi), \quad \xi \in \mathbb{R},$$

namely H_k is an eigenfunction of the one-dimensional Ornstein-Uhlenbeck operator, with eigenvalue $-k$. This property is extended to any dimension as follows.

Proposition 14.1.1. *For every $\alpha \in \Lambda$, H_α belongs to $D(L_2)$ and*

$$L_2 H_\alpha = -|\alpha| H_\alpha.$$

Proof. As a first step, we consider the finite dimensional case $X = \mathbb{R}^d$, $\gamma = \gamma_d$. Then $H = \mathbb{R}^d$ and we take the canonical basis of $\mathbb{R}^d$ as a basis for H, so that $\hat{h}_j(x) = x_j$ for $j = 1, \dots d$.

We fix a Hermite polynomial H_α in $\mathbb{R}^d$, with $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$,

$$H_\alpha(x) = \prod_{j=1}^{d} H_{\alpha_j}(x_j), \quad x \in \mathbb{R}^d.$$

H_α belongs to $W^{2,2}(\mathbb{R}^d, \gamma_d)$ (in fact, it belongs to $W^{2,p}(\mathbb{R}^d, \gamma_d)$ for every $p \in [1, +\infty)$) and therefore by Theorem 13.1.4, it is in $D(L_2^{(d)})$. By (8.1.4) we know that $L_2^{(d)} H_\alpha = \mathcal{L}^{(d)} H_\alpha = -|\alpha| H_\alpha$.

Now we turn to the infinite dimensional case. Let $\alpha \in \Lambda$ and let $d \in \mathbb{N}$ be such that $\alpha_j = 0$ for each $j > d$. Then $H_\alpha(x) = \varphi(\hat{h}_1(x), \dots, \hat{h}_d(x))$, where φ is a Hermite polynomial in $\mathbb{R}^d$. Proposition 13.2.1 implies that $H_\alpha \in D(L_2)$, and

$$L_2 H_\alpha = \mathcal{L}^{(d)} \varphi(\hat{h}_1(\cdot), \dots, \hat{h}_d(\cdot)) = -|\alpha| \varphi(\hat{h}_1(\cdot), \dots, \hat{h}_d(\cdot)) = -|\alpha| H_\alpha.$$

□

As a consequence of Propositions 14.1.1 and 8.1.9, we characterise the spectrum of L_2. We recall that, for every $k \in \mathbb{N} \cup \{0\}$, I_k is the orthogonal projection on the subspace $\mathcal{X}_k = \overline{\text{span}\{H_\alpha : \alpha \in \Lambda, |\alpha| = k\}}$ of $L^2(X, \gamma)$. See Section 8.1.2.

Proposition 14.1.2. *The spectrum of L_2 is equal to $-\mathbb{N} \cup \{0\}$. For every $k \in \mathbb{N} \cup \{0\}$, $\mathcal{X}_k$ is the eigenspace of L_2 with eigenvalue $-k$. Therefore, $I_k(L_2 f) = L_2(I_k f) = -kI_k(f)$, for every $f \in D(L_2)$.*

Proof. Let us consider the point spectrum. First of all, we prove that $\mathcal{X}_k$ is contained in the eigenspace of L_2 with eigenvalue $-k$.

$\mathcal{X}_0$ consists of constant functions, that belong to the kernel of L_2. For $k \in \mathbb{N}$, every element $f \in \mathcal{X}_k$ is equal to $\lim_{n\to\infty} f_n$, where each f_n is a linear combination of Hermite polynomials H_α with $|\alpha| = k$. By Proposition 14.1.1, $f_n \in D(L_2)$ and $L_2 f_n = -kf_n$. Since L_2 is a closed operator, $f \in D(L_2)$ and $L_2 f = -kf$.

Let now $f \in D(L_2)$ be such that $L_2 f = \lambda f$ for some $\lambda \in \mathbb{R}$. For every $\alpha \in \Lambda$ we have

$$\lambda \langle f, H_\alpha \rangle_{L^2(X,\gamma)} = \langle L_2 f, H_\alpha \rangle_{L^2(X,\gamma)} = \langle f, L_2 H_\alpha \rangle_{L^2(X,\gamma)} = -|\alpha| \langle f, H_\alpha \rangle_{L^2(X,\gamma)}.$$

Therefore, either $\lambda = -|\alpha|$ or $\langle f, H_\alpha\rangle_{L^2(X,\gamma)} = 0$. If $\lambda = -k$ with $k \in \mathbb{N}\cup\{0\}$, then f is orthogonal to all Hermite polynomials H_β with $|\beta| \neq k$, hence $f \in \mathcal{X}_k$ is an eigenfunction of L_2 with eigenvalue $-k$. If $\lambda \neq -k$ for every $k \in \mathbb{N}\cup\{0\}$, then f is orthogonal to all Hermite polynomials so that it vanishes. This proves that $\mathcal{X}_k$ is equal to the eigenspace of L_2 with eigenvalue $-k$.

Since L_2 is self-adjoint, for $f \in D(L_2)$ and $|\alpha| = k$ we have

$$\langle L_2 f, H_\alpha\rangle_{L^2(X,\gamma)} = \langle f, L_2 H_\alpha\rangle_{L^2(X,\gamma)} = -k\langle f, H_\alpha\rangle_{L^2(X,\gamma)}. \tag{14.1.1}$$

Let f_j, $j \in \mathbb{N}$, be any enumeration of the Hermite polynomials H_α with $|\alpha| = k$. The sequence $s_n := \sum_{j=0}^n \langle f, f_j\rangle_{L^2(X,\gamma)} f_j$ converges in $D(L_2)$, since $L^2 - \lim_{n\to\infty} s_n = I_k(f)$ and

$$L_2 s_n = \sum_{j=0}^n \langle f, f_j\rangle_{L^2(X,\gamma)} L_2 f_j = -k\sum_{j=0}^n \langle f, f_j\rangle_{L^2(X,\gamma)} f_j = \sum_{j=0}^n \langle L_2 f, f_j\rangle_{L^2(X,\gamma)} f_j,$$

where the last equality follows from (14.1.1). The series in the right hand side converges to $-kI_k(f) = I_k(L_2 f)$, as $n \to \infty$. Then, $L_2 I_k(f) = -kI_k(f) = I_k(L_2 f)$, for every $k \in \mathbb{N}\cup\{0\}$.

It remains to show that the spectrum of L_2 is just $-\mathbb{N}\cup\{0\}$. We notice that $D(L_2)$ is not compactly embedded in $L^2(X,\gamma)$ if X is infinite dimensional, because it has infinite dimensional eigenspaces. So, the spectrum does not necessarily consist of eigenvalues.

If $\lambda \neq -h$ for every $h \in \mathbb{N}\cup\{0\}$, and $f \in L^2(X,\gamma)$, the resolvent equation $\lambda u - L_2 u = f$ is equivalent to $\lambda I_k(u) - I_k(L_2 u) = I_k(f)$ for every $k \in \mathbb{N}\cup\{0\}$, and therefore to $\lambda I_k(u) + kI_k(u) = I_k(f)$, for every $k \in \mathbb{N}\cup\{0\}$. So, we define

$$u = \sum_{k=0}^\infty \frac{1}{\lambda + k} I_k(f). \tag{14.1.2}$$

The sequence $u_n := \sum_{k=0}^n I_k(f)/(\lambda+k)$ converges in $D(L_2)$, since both sequences $1/(\lambda+k)$ and $k/(\lambda+k)$ are bounded. Therefore, $u \in D(L_2)$, and $\lambda u - L_2 u = f$. □

Another consequence is a characterisation of L_2 in terms of Hermite polynomials.

Proposition 14.1.3.

$$\begin{cases} (a) \quad D(L_2) = \Big\{ f \in L^2(X,\gamma) : \ \displaystyle\sum_{k=1}^\infty k^2 \|I_k(f)\|^2_{L^2(X,\gamma)} < \infty \Big\}, \\ \\ (b) \quad L_2 f = -\displaystyle\sum_{k=1}^\infty k I_k(f), \quad f \in D(L_2). \end{cases} \tag{14.1.3}$$

Proof. Let $f \in D(L_2)$. Then $I_k(L_2 f) = -kI_k(f) = L_2(I_k(f))$ for every $k \in \mathbb{N}\cup\{0\}$, by Proposition 14.1.2. Applying (8.1.9) to $L_2 f$ we obtain

$$L_2 f = \sum_{k=0}^\infty I_k(L_2 f) = \sum_{k=1}^\infty -kI_k(f)$$

which proves (14.1.3)(b). Moreover,

$$\|L_2 f\|^2_{L^2(X,\gamma)} = \sum_{k=1}^{\infty} k^2 \|I_k(f)\|^2_{L^2(X,\gamma)} < \infty.$$

Conversely, let $f \in L^2(X,\gamma)$ be such that $\sum_{k=1}^{\infty} k^2 \|I_k(f)\|^2_{L^2(X,\gamma)} < \infty$. Then the sequence

$$f_n := \sum_{k=0}^{n} I_k(f)$$

converges to f in $L^2(X,\gamma)$, and it converges in $D(L_2)$ too, since for $n > m$

$$\|L_2(f_n - f_m)\|^2_{L^2(X,\gamma)} = \Big\| \sum_{k=m+1}^{n} -kI_k(f) \Big\|^2_{L^2(X,\gamma)} = \sum_{k=m+1}^{n} k^2 \|I_k(f)\|^2_{L^2(X,\gamma)} \to 0 \text{ as } m \to \infty.$$

Since L_2 is closed, $f \in D(L_2)$. □

As every H_α is an eigenfunction of L_2 with eigenvalue $-|\alpha|$, we ask to verify that

$$T_2(t)H_\alpha = e^{|\alpha|t} H_\alpha, \quad t \geq 0,\ \alpha \in \Lambda, \tag{14.1.4}$$

see Exercise 14.1. As a consequence, we obtain a very handy expression of $T_2(t)$ in terms of Hermite polynomials.

Corollary 14.1.4. *For every $t > 0$ we have*

$$T_2(t)f = \sum_{k=0}^{\infty} e^{-kt} I_k(f), \quad f \in L^2(X,\gamma), \tag{14.1.5}$$

where the series converges in $L^2(X,\gamma)$. Moreover, $T_2(t)f \in D(L_2)$ and

$$\|L_2 T_2(t) f\|_{L^2(X,\gamma)} \leq \frac{1}{te} \|f\|_{L^2(X,\gamma)}. \tag{14.1.6}$$

The function $t \mapsto T(t)f$ belongs to $C^1((0,+\infty); L^2(X,\gamma))$, and

$$\frac{d}{dt} T_2(t) f = L_2 T_2(t) f, \quad t > 0. \tag{14.1.7}$$

Proof. Fix $f \in L^2(X,\gamma)$. By Lemma 14.1.1, for every $k \in \mathbb{N}$, $I_k(f) \in D(L_2)$ and $L_2 I_k(f) = -kI_k(f)$, so that by the above considerations, $T_2(t)I_k(f) = e^{-kt} I_k(f)$. Since $f = \lim_{n\to\infty} \sum_{k=0}^{n} I_k(f)$ in $L^2(X,\gamma)$ and $T_2(t)$ is a bounded operator in $L^2(X,\gamma)$, (14.1.5) follows.

The other statements and estimate $\|L_2 T_2(t) f\|_{L^2(X,\gamma)} \leq c\|f\|_{L^2(X,\gamma)}/t$ follow from the fact that $T_2(t)$ is an analytic semigroup in $L^2(X,\gamma)$, see Theorem 11.4.2. However, we give here a simple independent proof, specifying the constant $c = 1/e$ in (14.1.6).

Since $\sup_{\xi>0}\xi^2 e^{-2\xi} = e^{-2}$, using (14.1.5) we obtain

$$\sum_{k=1}^\infty k^2\|I_k(T_2(t)f)\|^2_{L^2(X,\gamma)} = \sum_{k=1}^\infty k^2 e^{-2kt}\|I_k(f)\|^2_{L^2(X,\gamma)}$$
$$\leq \frac{1}{e^2t^2}\sum_{k=1}^\infty \|I_k(f)\|^2_{L^2(X,\gamma)} \leq \frac{1}{e^2t^2}\|f\|^2_{L^2(X,\gamma)}$$

so that $T_2(t)f \in D(L_2)$ by (14.1.3)(a), and estimate (14.1.6) follows from (14.1.3)(b). Moreover, for every $t>0$ and $0<|h|\leq t/2$ we have

$$\left\|\frac{1}{h}(T_2(t+h)f - T_2(t)f) - L_2T_2(t)f\right\|^2_{L^2(X,\gamma)} = \sum_{k=0}^\infty \left(\frac{e^{-k(t+h)}-e^{-kt}}{h}+k\right)^2\|I_k(f)\|^2_{L^2(X,\gamma)}.$$

Each addend in the right hand side sum converges to 0, and using the Taylor formula for the exponential function we easily obtain

$$\left(\frac{e^{-k(t+h)}-e^{-kt}}{h}+k\right)^2 \leq \frac{t^2}{4}k^4e^{-2kt} \leq \frac{c}{t^2},$$

with c independent of t, h, k. By the Dominated Convergence Theorem for series, we obtain

$$\lim_{h\to 0^+}\left\|\frac{1}{h}(T_2(t+h)f - T_2(t)f) - L_2T_2(t)f\right\|_{L^2(X,\gamma)} = 0,$$

namely, the function $T_2(\cdot)f$ is differentiable at t, with derivative $L_2T_2(t)f$. For $t>t_0>0$ we have $L_2T_2(t)f = L_2T_2(t-t_0)T(t_0)f = T_2(t-t_0)L_2T(t_0)f$. Then, $t\mapsto L_2T_2(t)f$ is continuous in $[t_0,+\infty)$. Since t_0 is arbitrary, $T(\cdot)f$ belongs to $C^1((0,+\infty);L^2(X,\gamma))$. □

We already know that $D(L_2) = W^{2,2}(X,\gamma)$. So, Proposition 14.1.3 gives a characterisation of $W^{2,2}(X,\gamma)$ in terms of Hermite polynomials. A similar characterisation is available for the space $W^{1,2}(X,\gamma)$.

Proposition 14.1.5.

$$W^{1,2}(X,\gamma) = \Big\{f\in L^2(X,\gamma):\ \sum_{k=1}^\infty k\|I_k(f)\|^2_{L^2(X,\gamma)}<\infty\Big\}.$$

Moreover, for every $f\in W^{1,2}(X,\gamma)$,

$$\int_X |\nabla_H f|^2_H\,d\gamma = \sum_{k=1}^\infty k\int_X (I_k(f))^2 d\gamma,$$

and

(i) *for every* $f\in W^{1,2}(X,\gamma)$ *the sequence* $\sum_{k=0}^n I_k(f)$ *converges to* f *in* $W^{1,2}(X,\gamma)$, *(ii) the sequence* $\sum_{k=1}^{n}\sqrt{k}I_k$ *converges in* $\mathcal{L}(W^{1,2}(X,\gamma),L^2(X,\gamma))$.

Proof. Let $f\in L^2(X,\gamma)$. By Proposition 14.1.2, for every $k\in\mathbb{N}$, $I_k(f)\in D(L_2)$ and $L_2I_k(f)=-kI_k(f)$. Therefore,

$$\int_X|\nabla_H I_k(f)|_H^2d\gamma=-\int_X I_k(f)L_2I_k(f)\,d\gamma=k\int_X(I_k(f))^2d\gamma,\quad k\in\mathbb{N}. \tag{14.1.8}$$

Assume that $\sum_{k=1}^{\infty}k\|I_k(f)\|^2_{L^2(X,\gamma)}<\infty$. The sequence $s_n:=\sum_{k=0}^{n}I_k(f)$ converges to f in $L^2(X,\gamma)$. Moreover, $(\nabla_H s_n)$ is a Cauchy sequence in $L^2(X,\gamma;H)$. Indeed, $\nabla_H I_k(f)$ and $\nabla_H I_l(f)$ are orthogonal in $L^2(X,\gamma;H)$ for $l\neq k$, because

$$\int_X[\nabla_H I_k(f),\nabla_H I_l(f)]_Hd\gamma=-\int_X I_k(f)L_2I_l(f)d\gamma=l\int_X I_k(f)I_l(f)d\gamma=0.$$

Therefore, for n, $p\in\mathbb{N}$,

$$\Big\|\sum_{k=n}^{n+p}\nabla_H I_k(f)\Big\|^2_{L^2(X,\gamma;H)}=\sum_{k=n}^{n+p}\int_X|\nabla_H I_k(f)|_H^2d\gamma=\sum_{k=n}^{n+p}k\|I_k(f)\|^2_{L^2(X,\gamma)}.$$

So, $f\in W^{1,2}(X,\gamma)$, $s_n\to f$ in $W^{1,2}(X,\gamma)$, and

$$\int_X|\nabla_H f|_H^2d\gamma=\sum_{k=1}^{\infty}\int_X|\nabla_H I_k(f)|_H^2d\gamma=\sum_{k=1}^{\infty}k\|I_k(f)\|^2_{L^2(X,\gamma)}.$$

To prove the converse, first we take $f\in D(L_2)$. Then, by (13.2.5),

$$\begin{aligned}\int_X|\nabla_H f|_H^2d\gamma&=-\int_X fL_2f\,d\gamma=-\int_X\sum_{l=0}^{\infty}I_l(f)\sum_{k=0}^{\infty}I_k(L_2f)\,d\gamma\\&=-\int_X\sum_{k=0}^{\infty}I_k(f)I_k(L_2f)\,d\gamma,\end{aligned}$$

since $I_l(f)\in\mathfrak{X}_l$, $I_k(L_2f)\in\mathfrak{X}_k$. By Proposition 14.1.2,

$$\int_X|\nabla_H f|_H^2d\gamma=\sum_{k=1}^{\infty}k\int_X(I_k(f))^2d\gamma.$$

Comparing with (14.1.8), we obtain

$$\int_X|\nabla_H f|_H^2d\gamma=\sum_{k=1}^{\infty}\int_X|\nabla_H I_k(f)|_H^2d\gamma.$$

So, the mappings $T_n:D(L_2)\to L^2(X,\gamma)$, $T_nf=\sum_{k=1}^{n}\sqrt{k}I_k(f)$ satisfy

$$\exists L^2(X,\gamma)-\lim_{n\to\infty}T_nf,\quad\|T_nf\|_{L^2(X,\gamma)}\le\|f\|_{W^{1,2}(X,\gamma)}.$$

Since $D(L_2)$ is dense in $W^{1,2}(X,\gamma)$, the sequence (T_nf) converges in $L^2(X,\gamma)$ for every $f\in W^{1,2}(X,\gamma)$. Since $\|T_nf\|^2_{L^2(X,\gamma)}=\sum_{k=1}^{n}k\|I_k(f)\|^2_{L^2(X,\gamma)}$, letting $n\to\infty$ we get $\sum_{k=1}^{\infty}k\|I_k(f)\|^2_{L^2(X,\gamma)}<\infty$. □

Proposition 14.1.3 may be recognized as the spectral decomposition of L. See e.g. [RS, §VIII.3], in particular Theorem VIII.6. Accordingly, a functional calculus for L may be defined , namely for every $g : -\mathbb{N}\cup\{0\} \to \mathbb{R}$ we set

$$D(g(L)) := \Big\{ f \in L^2(X,\gamma) : \sum_{k=0}^{\infty} |g(-k)|^2 \|I_k(f)\|^2_{L^2(X,\gamma)} < \infty \Big\}$$

and

$$g(L)(f) = \sum_{k=0}^{\infty} g(-k) I_k(f), \quad f \in D(g(L)).$$

In particular, for $g(\xi) = (-\xi)^{1/2}$, Proposition 14.1.5 says that $D((-L)^{1/2}) = W^{1,2}(X,\gamma)$, and $\|(-L)^{1/2} f\|_{L^2(X,\gamma)} = \| \, |\nabla_H f|_H \|_{L^2(X,\gamma)}$ for every $f \in W^{1,2}(X,\gamma)$.

Corollary 14.1.6. *For every $d \in \mathbb{N}$, the embedding $W^{1,2}(\mathbb{R}^d,\gamma_d) \subset L^2(\mathbb{R}^d,\gamma_d)$ is compact.*

Proof. Let (f_n) be a bounded sequence in $W^{1,2}(\mathbb{R}^d,\gamma_d)$, say $\|f_n\|_{W^{1,2}(\mathbb{R}^d,\gamma_d)} \leq C$. Then there exists a subsequence (f_{n_j}) that converges weakly in $W^{1,2}(\mathbb{R}^d,\gamma_d)$ to an element $f \in W^{1,2}(\mathbb{R}^d,\gamma_d)$, that still satisfies $\|f\|_{W^{1,2}(\mathbb{R}^d,\gamma_d)} \leq C$. We claim that $f_{n_j} \to f$ in $L^2(\mathbb{R}^d,\gamma_d)$.

For every $N \in \mathbb{N}$ we have (norms and inner products are in $L^2(\mathbb{R}^d,\gamma_d)$)

$$\begin{aligned}
\|f_{n_j} - f\|^2 &= \sum_{k=0}^{\infty} \|I_k(f_{n_j} - f)\|^2 = \sum_{k=0}^{N-1} \|I_k(f_{n_j} - f)\|^2 + \sum_{k=N}^{\infty} \|I_k(f_{n_j} - f)\|^2 \\
&\leq \sum_{k=0}^{N-1} \sum_{\alpha\in(\mathbb{N}\cup\{0\})^d,\, |\alpha|=k} \langle f_{n_j} - f, H_\alpha\rangle^2 + \frac{1}{N}\sum_{k=N}^{\infty} k\|I_k(f_{n_j} - f)\|^2 \\
&\leq \sum_{k=0}^{N-1} \sum_{\alpha\in(\mathbb{N}\cup\{0\})^d,\, |\alpha|=k} \langle f_{n_j} - f, H_\alpha\rangle^2 + \frac{(2C)^2}{N}.
\end{aligned}$$

Fixed any $\varepsilon > 0$, let N be such that $4C^2/N \leq \varepsilon$. The sum in the right hand side consists of a finite number of summands, each of them goes to 0 as $n_j \to \infty$, therefore it does not exceed ε provided n_j is large enough. □

The argument in the proof of Corollary 14.1.6 does not work in infinite dimension, because in this case for every $k \in \mathbb{N}$ the Hermite polynomials H_α with $|\alpha| = k$ are infinitely many. In fact, $W^{1,2}(X,\gamma)$ is not compactly embedded in $L^2(X,\gamma)$ if H is infinite dimensional. It is sufficient to consider the Hermite polynomials H_α with $|\alpha| = 1$, namely the sequence of functions $(\hat{h}_j)$. Their $W^{1,2}(X,\gamma)$ norm is 2 but no subsequence converges in $L^2(X,\gamma)$ since $\|\hat{h}_i - \hat{h}_j\|^2_{L^2(X,\gamma)} = 2$ for $i \neq j$. The same argument shows that $D(L_2)$ is not compactly embedded in $L^2(X,\gamma)$.

14.2 Functional inequalities and asymptotic behaviour

In this section we present two important inequalities, the Logarithmic Sobolev and Poincaré inequality, that hold for functions in Sobolev spaces. The Ornstein-Uhlenbeck semigroup can be used as a tool in their proofs, and, in their turn, they are used to prove summability improving and asymptotic behaviour results for $T_p(t)f$, as $t \to \infty$.

We introduce the mean value $\overline{f}$ of any $f \in L^1(X, \gamma)$,

$$\overline{f} := \int_X f \, d\gamma.$$

If $f \in L^2(X, \gamma)$, $\overline{f} = I_0(f)$ is just the orthogonal projection of f on the kernel $\mathcal{X}_0$ of L_2, that consists of constant functions by Proposition 14.1.2 (see also Exercise 13.4). In any case, we have the following asymptotic behavior result.

Lemma 14.2.1. *For every $f \in C_b(X)$,*

$$\lim_{t\to+\infty} T(t)f(x) = \overline{f}, \quad x \in X. \tag{14.2.1}$$

For every $f \in L^p(X, \gamma)$, $1 \le p < \infty$,

$$\lim_{t\to+\infty} \|T_p(t)f - \overline{f}\|_{L^p(X,\gamma)} = 0. \tag{14.2.2}$$

Proof. The first assertion is an easy consequence of the definition (12.1.1) of $T(t)f$, through the Dominated Convergence Theorem. Still for $f \in C_b(X)$, we have that (14.2.2) holds again by the Dominated Convergence Theorem. Since $C_b(X)$ is dense in $L^p(X, \gamma)$ and the linear operators $f \mapsto T_p(t)f - \overline{f}$ belong to $\mathcal{L}(L^p(X, \gamma))$ and have norm not exeeding 2, the second assertion follows as well. □

We shall see that the rate of convergence in (14.2.2) is exponential. This fact could be seen as a consequence of general results on analytic semigroups, but here we shall give a simpler and direct independent proof.

14.2.1 The Logarithmic Sobolev inequality

To begin with, we remark that no Sobolev embedding holds for nondegenerate Gaussian measures. Even in dimension 1, the function

$$f(\xi) = \frac{e^{\xi^2/4}}{1 + \xi^2}, \quad \xi \in \mathbb{R},$$

belongs to $W^{1,2}(\mathbb{R}, \gamma_1)$ but it does not belong to $L^{2+\varepsilon}(\mathbb{R}, \gamma_1)$ for any $\varepsilon > 0$. This example may be adapted to show that for every $p \ge 1$, $W^{1,p}(\mathbb{R}, \gamma_1)$ is not contained in $L^{p+\varepsilon}(\mathbb{R}, \gamma_1)$ for any $\varepsilon > 0$, see Exercise 14.2.

The best result about summability properties in this context is the next Logarithmic Sobolev (Log-Sobolev) inequality. In the following we set $0 \log 0 = 0$.

Theorem 14.2.2. *Let $p > 1$. For every $f \in C_b^1(X)$ we have*

$$\int_X |f|^p \log |f|\, d\gamma \leq \|f\|^p_{L^p(X,\gamma)} \log \|f\|_{L^p(X,\gamma)} + \frac{p}{2}\int_X |f|^{p-2}|\nabla_H f|_H^2 1\!\!1_{\{f\neq 0\}} d\gamma. \tag{14.2.3}$$

Proof. As a first step, we consider a function f with positive infimum, say $f(x) \geq c > 0$ for every x. In this case, also f^p belongs to $C_b^1(X)$, and $(T(t)f^p)(x) \geq c^p$ for every x, by (12.1.1). We define the function

$$F(t) = \int_X (T(t)f^p) \log(T(t)f^p) d\gamma, \quad t \geq 0.$$

Since L_2 is a sectorial operator (or, by Corollary 14.1.4), the function $t \mapsto T(t)f^p$ and $t \mapsto \log(T(t)f^p)$ belong to $C^1((0,+\infty); L^2(X,\gamma))$. Consequently, their product is in $C^1((0,+\infty); L^1(X,\gamma))$, $F \in C^1(0,+\infty)$, and for every $t > 0$ we have

$$\begin{aligned} F'(t) &= \int_X [L_2(T(t)f^p)\cdot \log(T(t)f^p) + L_2T(t)f^p]\, d\gamma \qquad (14.2.4)\\ &= \int_X L_2(T(t)f^p)\cdot \log(T(t)f^p)\, d\gamma. \end{aligned}$$

The second equality is a consequence of the invariance of γ (Propositions 12.1.5(iii) and 11.3.1). Moreover, $t \mapsto T(t)f^p(x)$ and $t \mapsto \log(T(t)f^p)(x)$ are continuous for every x and bounded by constants independent of x. It follows that F is continuous up to $t = 0$, and $F(t) - F(0) = \int_0^t F'(s)ds$. Integrating in the right hand side of (14.2.4) and using (13.2.5) with f replaced by $T(t)f^p$, g replaced by $\log(T(t)f^p)$, we obtain

$$\begin{aligned} F'(t) &= -\int_X [\nabla_H T(s)f^p, \nabla_H \log(T(s)f^p))]_H d\gamma \\ &= -\int_X \frac{1}{T(t)f^p}|\nabla_H(T(t)f^p)|_H^2\, d\gamma. \end{aligned}$$

We recall that for every $x \in X$, $|\nabla_H(T(t)f^p)(x)|_H \leq e^{-t}T(t)(|\nabla_H f^p|_H)(x)$ (see Proposition 12.1.6). So,

$$F'(t) \geq -e^{-2t}\int_X \frac{1}{T(t)f^p}(T(t)(|\nabla_H f^p|_H))^2\, d\gamma. \tag{14.2.5}$$

Moreover, using the Hölder inequality in (12.1.1) yields

$$|T(t)(\varphi_1\varphi_2)(x)| \leq [(T(t)\varphi_1^2)(x)]^{1/2}[(T(t)\varphi_2^2)(x)]^{1/2}, \quad \varphi_i \in C_b(X),\ x \in X.$$

We use this estimate with $\varphi_1 = |\nabla_H f^p|_H/f^{p/2}$, $\varphi_2 = f^{p/2}$ and we obtain

$$T(t)(|\nabla_H f^p|_H) = T(t)\Big(\frac{|\nabla_H f^p|_H}{f^{p/2}}\, f^{p/2}\Big) \leq \Big(T(t)\Big(\frac{|\nabla_H f^p|_H^2}{f^p}\Big)\Big)^{1/2}(T(t)f^p)^{1/2}.$$

Replacing in (14.2.5) and using (12.1.2), we get

$$\begin{aligned} F'(t) &\geq -e^{-2t}\int_X T(t)\Big(\frac{|\nabla_H f^p|_H^2}{f^p}\Big)d\gamma = -e^{-2t}\int_X \frac{|\nabla_H f^p|_H^2}{f^p}\, d\gamma \\ &= -p^2e^{-2t}\int_X f^{p-2}|\nabla_H f|_H^2 d\gamma. \end{aligned}$$

Integrating with respect to time in $(0,t)$ yields

$$\begin{aligned}\int_X (T(t)f^p)\log(T(t)f^p)\,d\gamma - \int_X f^p\log(f^p)d\gamma &= F(t)-F(0)\\ &\geq \frac{p^2}{2}(e^{-2t}-1)\int_X f^{p-2}|\nabla_H f|_H^2 d\gamma.\end{aligned} \tag{14.2.6}$$

Now we let $t\to+\infty$. By Lemma 14.2.1, $\lim_{t\to+\infty}(T(t)f^p)(x) = \overline{f^p} = \|f\|_{L^p}^p$, and consequently $\lim_{t\to+\infty}\log((T(t)f^p)(x)) = p\log(\|f\|_{L^p})$, for every $x\in X$. Moreover, $c^p \leq |(T(t)f^p)(x)| \leq \|f\|_\infty^p$, for every x. By the Dominated Convergence Theorem, the left hand side of (14.2.6) converges to $p\|f\|_{L^p}^p\log(\|f\|_{L^p}) - p\int_X f^p\log f\,d\gamma$ as $t\to+\infty$, and (14.2.3) follows.

For $f\in C_b^1(X)$ we approximate $|f|$ in $W^{1,p}(X,\gamma)$ and pointwise by the sequence $f_n=\sqrt{f^2+1/n}$, see Exercise 14.3. Applying (14.2.3) to each f_n we get

$$\begin{aligned}\int_X f_n^p\log f_n\,d\gamma - \|f_n\|_{L^p(X,\gamma)}^p\log\|f_n\|_{L^p(X,\gamma)} &\leq \frac{p}{2}\int_X f^2(f^2+1/n)^{p/2-2}|\nabla_H f|_H^2 d\gamma\\ &\leq \frac{p}{2}\int_X 1\!\!1_{\{f\neq 0\}}(f^2+1/n)^{p/2-1}|\nabla_H f|_H^2 d\gamma,\end{aligned}$$

and letting $n\to\infty$ yields that f satisfies (14.2.3). Notice that the last integral goes to $\int_X 1\!\!1_{\{f\neq 0\}}|f|^{p-2}|\nabla_H f|_H^2 d\gamma$ by the Monotone Convergence Theorem, even if $p<2$. □

Corollary 14.2.3. *Let $p\geq 2$. For every $f\in W^{1,p}(X,\gamma)$ we have*

$$\int_X |f|^p\log|f|\,d\gamma \leq \|f\|_{L^p(X,\gamma)}^p\log\|f\|_{L^p(X,\gamma)} + \frac{p}{2}\int_X |f|^{p-2}|\nabla_H f|_H^2 d\gamma. \tag{14.2.7}$$

Proof. We approximate f by a sequence of $\mathcal{F}C_b^1(X)$ functions (f_n) that converges in $W^{1,p}(X,\gamma)$ and pointwise a.e. to f. We apply (14.2.3) to each f_n, and then we let $n\to\infty$. Recalling that $\nabla_H f_n=0$ a.e. in the set $\{f_n=0\}$ (see Exercise 10.3), we get

$$\int_X |f_n|^{p-2}|\nabla_H f_n|_H^2 1\!\!1_{\{f_n\neq 0\}}\,d\gamma = \int_X |f_n|^{p-2}|\nabla_H f_n|_H^2\,d\gamma$$

for every n, and

$$\begin{aligned}\int_X |f|^p\log|f|\,d\gamma &\leq \liminf_{n\to\infty}\int_X |f_n|^p\log|f_n|\,d\gamma\\ &\leq \liminf_{n\to\infty}\left(\|f_n\|_{L^p(X,\gamma)}^p\log\|f_n\|_{L^p(X,\gamma)} + \frac{p}{2}\int_X |f_n|^{p-2}|\nabla_H f_n|_H^2 d\gamma\right)\\ &= \|f\|_{L^p(X,\gamma)}^p\log\|f\|_{L^p(X,\gamma)} + \frac{p}{2}\int_X |f|^{p-2}|\nabla_H f|_H^2 d\gamma\end{aligned}$$

□

Note that for $1 < p < 2$ the function $1\!\!1_{f\neq 0}|f|^{p-2}|\nabla_H f|_H^2$ does not necessarily belong to $L^1(X,\gamma)$ for $f \in W^{1,p}(X,\gamma)$, and in this case (14.2.3) is not meaningful, since it says that the left hand side does not exceed $+\infty$. Take for instance $X = \mathbb{R}$ and $f(x) = x^{1/p}$ for $0 < x < 1$, $f(x) = 0$ for $x \leq 0$, $f(x) = 1$ for $x \geq 1$. Then $f \in W^{1,p}(\mathbb{R},\gamma_1)$ but $\int_{\mathbb{R}} |f|^{p-2}|\nabla_H f|_H^2 1\!\!1_{\{f\neq 0\}} d\gamma_1 = +\infty$.

Instead, it is possible to show that for any $p \in (1,+\infty)$

$$-(p-1)\int_X |f|^{p-2}|\nabla_H f|_H^2 1\!\!1_{\{f\neq 0\}} d\gamma = \int_X f|f|^{p-2} L_p f\, d\gamma \tag{14.2.8}$$

for every $f \in D(L_p)$, so that $\int_X |f|^{p-2}|\nabla_H f|_H^2 1\!\!1_{\{f\neq 0\}} d\gamma \leq C_p\|f\|_{D(L_p)}$. See Exercise 14.4. So, if $f \in D(L_p)$ (14.2.3) may be rewritten as

$$\int_X |f|^p \log|f|\, d\gamma \leq \|f\|^p_{L^p(X,\gamma)} \log\|f\|_{L^p(X,\gamma)} - \frac{p}{2(p-1)}\int_X f|f|^{p-2} L_p f\, d\gamma. \tag{14.2.9}$$

An important consequence of the Log-Sobolev inequality is the next summability improving property of $T(t)$, called *hypercontractivity*.

Theorem 14.2.4. *Let $p > 1$, and set $p(t) = e^{2t}(p-1)+1$ for $t > 0$. Then $T_p(t)f \in L^{p(t)}(X,\gamma)$ for every $f \in L^p(X,\gamma)$, and*

$$\|T_p(t)f\|_{L^{p(t)}(X,\gamma)} \leq \|f\|_{L^p(X,\gamma)}, \quad t > 0. \tag{14.2.10}$$

Proof. Let us prove that (14.2.10) holds for every $f \in \Sigma$ with positive infimum (the set Σ was introduced at the beginning of Section 13.2, and it is dense in $L^p(X,\gamma)$). For such f's, since they belong to $D(L_q)$ for any q, we have that $T_p(f) = T(t)f$ and we can drop the idex p in the semigroup. We shall show that the function

$$\beta(t) := \|T(t)f\|_{L^{p(t)}(X,\gamma)}, \quad t \geq 0$$

decreases in $[0,+\infty)$.

It is easily seen that β is continuous in $[0,+\infty)$. Our aim is to show that $\beta \in C^1(0,+\infty)$, and $\beta'(t) \leq 0$ for every $t > 0$. Indeed, by Proposition 13.2.1 we know that for every $x \in X$ the function $t \mapsto T(t)f(x)$ belongs to $C^1(0,+\infty)$, as well as $t \mapsto (T(t)f(x))^{p(t)}$, and

$$\begin{aligned}\frac{d}{dt}(T(t)f(x))^{p(t)} &= p'(t)(T(t)f(x))^{p(t)}\log(T(t)f(x)) + p(t)(T(t)f(x))^{p(t)-1}\frac{d}{dt}(T(t)f(x))\\ &= p'(t)(T(t)f(x))^{p(t)}\log(T(t)f(x)) + p(t)(T(t)f(x))^{p(t)-1}(L_2 T(t)f(x)).\end{aligned}$$

We have used the operator L_2, but any other L_q can be equivalently used. Moreover, $|d/dt(T(t)f(x))^{p(t)}|$ is bounded by $c(t)(1+\|x\|)$ for some continuous function $c(\cdot)$. So, $t \mapsto \int_X |T(t)f|^{p(t)} d\gamma$ is continuously differentiable, with derivative equal to

$$p'(t)\int_X (T(t)f)^{p(t)}\log(T(t)f)d\gamma - p(t)(p(t)-1)\int_X T(t)f)^{p(t)-2}|\nabla_H T(t)f|_H^2\, d\gamma.$$

It follows that β is differentiable and

$$\beta'(t) = \beta(t)\Big[-\frac{p'(t)}{p(t)^2}\log\int_X (T(t)f)^{p(t)}d\gamma + +\frac{p'(t)}{p(t)}\frac{\int_X (T(t)f)^{p(t)}\log(T(t)f)d\gamma}{\int_X (T(t)f)^{p(t)}d\gamma}$$
$$- (p(t)-1)\frac{\int_X (T(t)f)^{p(t)-2}|\nabla_H T(t)f|_H^2\,d\gamma}{\int_X (T(t)f)^{p(t)}d\gamma}\Big].$$

The Logarithmic Sobolev inequality (14.2.3) yields

$$\int_X (T(t)f)^{p(t)}\log(T(t)f)d\gamma \le$$
$$\le \frac{1}{p(t)}\int_X (T(t)f)^{p(t)}d\gamma\log\int_X (T(t)f)^{p(t)}d\gamma + \frac{p(t)}{2}\int_X (T(t)f)^{p(t)-2}|\nabla_H T(t)f|_H^2\,d\gamma,$$

and replacing we obtain

$$\beta'(t) \le \left(\frac{p'(t)}{2} - (p(t)-1)\right)\frac{\int_X (T(t)f)^{p(t)-2}|\nabla_H T(t)f|_H^2\,d\gamma}{\int_X (T(t)f)^{p(t)}d\gamma}.$$

The function $p(t)$ was chosen in such a way that $p'(t) = 2(p(t)-1)$. Therefore, $\beta'(t) \le 0$, and (14.2.10) follows.

Let now $f \in \Sigma$ and set $f_n = (f^2 + 1/n)^{1/2}$. For every $x \in X$ and $n \in \mathbb{N}$ we have, by (12.1.1), $|(T(t)f)(x)| \le (T(t)|f|)(x) \le (T(t)f_n)(x)$, so that

$$\|T(t)f\|_{L^{p(t)}(X,\gamma)} \le \liminf_{n\to\infty}\|T(t)f_n\|_{L^{p(t)}(X,\gamma)} \le \liminf_{n\to\infty}\|f_n\|_{L^p(X,\gamma)} = \|f\|_{L^p(X,\gamma)},$$

and (14.2.10) holds. Since Σ is dense in $L^p(X,\gamma)$, (14.2.10) holds for every $f \in L^p(X,\gamma)$. □

We notice that in the proof of Theorem 14.2.4 we have not used specific properties of the Ornstein-Uhlenbeck semigroup: the main ingredients were the integration by parts formula, namely the fact that the infinitesimal generator L_2 is the operator associated to the quadratic form (13.2.4), and the Log-Sobolev inequality (14.2.3) for good functions. In fact, the proof may be extended to a large class of semigroups in spaces $L^p(\Omega,\mu)$, (Ω,μ) being a probability space, see [G]. In [G] a sort of converse is proved, namely under suitable assumptions if a semigroup $T(t)$ is a contraction from $L^p(\Omega,\mu)$ to $L^{q(t)}(\Omega,\mu)$, with q differentiable and increasing, then a logarithmic Sobolev inequality of the type (14.2.9) holds in the domain of the infinitesimal generator of $T(t)$ in $L^p(X,\mu)$.

14.2.2 The Poincaré inequality and the asymptotic behaviour

The Poincaré inequality is the following.

Theorem 14.2.5. *For every $f \in W^{1,2}(X,\gamma)$,*

$$\int_X (f-\overline{f})^2 d\gamma \le \int_X |\nabla_H f|_H^2 d\gamma. \qquad (14.2.11)$$

Proof. There are several proofs of (14.2.11). One of them follows from Theorem 14.2.4, see Exercise 14.5. The simplest proof uses the Wiener Chaos decomposition. By (8.1.9) and (14.1.3), for every $f \in D(L_2)$ we have $f = \sum_{k=0}^{\infty} I_k(f)$ and $L_2 f = \sum_{k=1}^{\infty} -kI_k(f)$, where both series converge in $L^2(X,\gamma)$. Using (13.2.5) and these representation formulas we obtain

$$\begin{aligned}\int_X |\nabla_H f|_H^2\, d\gamma &= -\int_X f\, L_2 f\, d\gamma \\ &= \sum_{k=1}^{\infty} k\|I_k(f)\|^2_{L^2(X,\gamma)} \geq \sum_{k=1}^{\infty} \|I_k(f)\|^2_{L^2(X,\gamma)} \\ &= \|f\|^2_{L^2(X,\gamma)} - \|I_0(f)\|^2_{L^2(X,\gamma)} \\ &= \|f\|^2_{L^2(X,\gamma)} - \overline{f}^2 = \|f-\overline{f}\|^2_{L^2(X,\gamma)}.\end{aligned}$$

Since $D(L_2)$ is dense in $W^{1,2}(X,\gamma)$, (14.2.11) follows. □

An immediate consequence of the Poincaré inequality is the following: if $f \in W^{1,2}(X,\gamma)$ and $\nabla_H f \equiv 0$, then f is constant a.e. (compare with Exercise 13.4).

An L^p version of (14.2.11) is

$$\int_X |f-\overline{f}|^p \gamma \leq c_p \int_X |\nabla_H f|_H^p d\gamma. \tag{14.2.12}$$

that holds for $p > 2$, $f \in W^{1,p}(X,\gamma)$ (Exercise 14.6).

Now we are able to improve Lemma 14.2.1, specifying the decay rate of $T_q(t)f$ to $\overline{f}$.

Proposition 14.2.6. *For every $q > 1$ there exists $c_q > 0$ such that $c_2 = 1$ and for every $f \in L^q(X,\gamma)$,*

$$\|T_q(t)f - \overline{f}\|_{L^q(X,\gamma)} \leq c_q e^{-t} \|f\|_{L^q(X,\gamma)}, \quad t > 0. \tag{14.2.13}$$

Proof. As a first step, we prove that the statement holds for $q = 2$. By (14.1.5), for every $f \in L^2(X,\gamma)$ and $t > 0$ we have $T(t)f = \sum_{k=0}^{\infty} e^{-kt} I_k(f)$. We already know that for $k = 0$, $I_0(f) = \overline{f}$. Therefore,

$$\|T(t)f - \overline{f}\|^2_{L^2(X,\gamma)} = \left\| \sum_{k=1}^{\infty} e^{-kt} I_k(f) \right\|^2_{L^2(X,\gamma)} \leq e^{-2t} \sum_{k=1}^{\infty} \|I_k(f)\|^2_{L^2(X,\gamma)} \leq e^{-2t} \|f\|^2_{L^2(X,\gamma)}.$$

For $q \neq 2$ it is enough to prove that (14.2.13) holds for every $f \in C_b(X)$. For such functions we have $T_q(t)f = T(t)f$ for every $t > 0$.

Let $q > 2$. Set $\tau = \log\sqrt{q-1}$, so that $e^{2\tau} + 1 = q$, and by Theorem 14.2.4 $T_q(\tau)$ is a contraction from $L^2(X,\gamma)$ to $L^q(X,\gamma)$. Then, for every $t \geq \tau$,

$$\begin{aligned}\|T(t)f - \overline{f}\|_{L^q(X,\gamma)} &= \|T(\tau)(T(t-\tau)f - \overline{f})\|_{L^q(X,\gamma)} \\ &\leq \|T(t-\tau)f - \overline{f}\|_{L^2(X,\gamma)} \quad \text{(by (14.2.10))} \\ &\leq e^{-(t-\tau)} \|f\|_{L^2(X,\gamma)} \quad \text{(by (14.2.13) with } q=2) \\ &\leq e^{-(t-\tau)} \|f\|_{L^q(X,\gamma)} \quad \text{(by the Hölder inequality)} \\ &= \sqrt{q-1}\, e^{-t} \|f\|_{L^q(X,\gamma)},\end{aligned}$$

while for $t \in (0,\tau)$ we have

$$\|T(t)f - \overline{f}\|_{L^q(X,\gamma)} \le 2\|f\|_{L^q(X,\gamma)} = 2e^t e^{-t}\|f\|_{L^q(X,\gamma)} \le 2\sqrt{q-1}\, e^{-t}\|f\|_{L^q(X,\gamma)}.$$

So, (14.2.13) holds with $c_q = 2\sqrt{q-1}$.

Let now $q < 2$ and set $\tau = -\log\sqrt{q-1}$, so that $e^{2\tau}(q-1)+1 = 2$, and by Theorem 14.2.4 $T_q(\tau)$ is a contraction from $L^q(X,\gamma)$ to $L^2(X,\gamma)$. For every $t \ge \tau$ we have

$$\begin{aligned}\|T(t)f - \overline{f}\|_{L^q(X,\gamma)} &\le \|T(t)f - \overline{f}\|_{L^2(X,\gamma)} \quad \text{(by the Hölder inequality)}\\ &= \|T(t-\tau)(T(\tau)f - \overline{T(\tau)f})\|_{L^2(X,\gamma)}\\ &\le e^{-(t-\tau)}\|T(\tau)f\|_{L^2(X,\gamma)} \quad \text{(by (14.2.13) with } q=2)\\ &\le e^{-(t-\tau)}\|f\|_{L^q(X,\gamma)} \quad \text{(by (14.2.10))}\\ &= \frac{1}{\sqrt{q-1}}\, e^{-t}\|f\|_{L^q(X,\gamma)},\end{aligned}$$

while for $t \in (0,\tau)$ we have, as before,

$$\|T(t)f - \overline{f}\|_{L^q(X,\gamma)} \le 2\|f\|_{L^q(X,\gamma)} = 2e^t e^{-t}\|f\|_{L^q(X,\gamma)} \le \frac{2}{\sqrt{q-1}}\, e^{-t}\|f\|_{L^q(X,\gamma)}.$$

So, (14.2.13) holds with $c_q = 2/\sqrt{q-1}$. □

In fact, estimate (14.2.13) could be deduced also by the general theory of (analytic) semigroups, but we prefer to give a simpler self-contained proof.

14.3 Exercises

Exercise 14.1. Prove the equality (14.1.4).

Exercise 14.2. Show that for every $p \ge 1$, $W^{1,p}(\mathbb{R},\gamma_1)$ is not contained in $L^{p+\varepsilon}(\mathbb{R},\gamma_1)$ for any $\varepsilon > 0$.

Exercise 14.3. Prove that for every $f \in W^{1,p}(X,\gamma)$ the sequence $f_n = \sqrt{f^2 + 1/n}$ converges to $|f|$ in $W^{1,p}(X,\gamma)$.

Exercise 14.4. Prove that for every $p > 1$ and $f \in D(L_p)$, (14.2.8) holds.
Hint: for every $f \in \Sigma$ and $\varepsilon > 0$, apply formula (13.2.5) with $g = f(f^2+\varepsilon)^{1-p/2}$ and then let $\varepsilon \to 0$.

Exercise 14.5. Prove the Poincaré inequality (14.2.11) for functions $f \in C^1_b(X)$ such that $\overline{f} = 0$, in the following alternative way: apply (14.2.7) with $p = 2$ to the functions $f_\varepsilon := 1 + \varepsilon f$, for $\varepsilon > 0$, and then divide by ε^2 and let $\varepsilon \to 0$.

Exercise 14.6. Prove that (14.2.12) holds for every $f \in W^{1,p}(X,\gamma)$ with $p > 2$.
Hint: For $p \le 4$, apply (14.2.11) to $|f|^{p/2}$ and estimate $(\int_X |f|^{p/2} d\gamma)^2$ by $\|f\|^p_{L^2(X,\gamma)}$, then

estimate $(\int_X |\nabla_H f|_H^2 |f|^{p/2-1} d\gamma)^2$ by $\varepsilon \int_X |f|^p d\gamma + C(\varepsilon)(\int_X |\nabla_H f|_H^p d\gamma)$. Taking ε small, arrive at

$$\int_X |f|^p d\gamma \le \|f\|_{L^2(X,\gamma)}^p + K \int_X |\nabla_H f|_H^p d\gamma.$$

(14.2.12) follows applying such estimate to $f - \overline{f}$, and using (14.2.11) to estimate $\|f - \overline{f}\|_{L^2(X,\gamma)}$. For $p \ge 4$, use a bootstrap procedure.

Bibliography

[B] V. I. Bogachev: *Gaussian Measures.* American Mathematical Society, 1998.

[G] L. Gross: *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), 1061-1083.

[RS] M. Reed, B. Simon: *Methods of modern mathematical physics. I: Functional analysis*, Academic Press, Inc., 1980.