Lecture 13

The Ornstein-Uhlenbeck operator

In this lecture we study the infinitesimal generator of $T_p(t)$, for $p \in (1, +\infty)$. The strongest result is the characterisation of the domain of the generator L_2 of $T_2(t)$ as the Sobolev space $W^{2,2}(X, \gamma)$. A similar result holds for $p \in (1, +\infty) \setminus \{2\}$, but the proof is much more complicated and will not be given here.

13.1 The finite dimensional case

Here, $X = \mathbb{R}^d$ and $\gamma = \gamma_d$. We describe the infinitesimal generator L_p of $T_p(t)$ in $L^p(\mathbb{R}^d, \gamma_d)$, for $p \in (1, +\infty)$, which is a suitable realisation of the Ornstein-Uhlenbeck differential operator

$$\mathcal{L} f(x) := \Delta f(x) - x \cdot \nabla f(x) \tag{13.1.1}$$

in $L^p(\mathbb{R}^d, \gamma_d)$.

We recall that

$$D(L_p) = \left\{ f \in L^p(\mathbb{R}^d, \gamma_d) : \exists L^p - \lim_{t \to 0^+} \frac{T(t)f - f}{t} \right\},$$
$$L_p f = \lim_{t \to 0^+} \frac{T(t)f - f}{t}.$$

If $f \in D(L_p)$, by Lemma 11.1.7 the function $t \mapsto T(t)f$ belongs to $C^1([0, +\infty); L^p(\mathbb{R}^d, \gamma_d)) \cap C([0, +\infty); D(L_p))$ and $d/dt\, T(t)f = L_p T(t)f$, for every $t \geq 0$. To find an expression of L_p, we differentiate $T(t)f$ with respect to time for good f. We recall that for $f \in C_b(\mathbb{R}^d)$, $T_p(t)f = T(t)f$ is given by formula (12.1.1).

Lemma 13.1.1. *For every $f \in C_b(\mathbb{R}^d)$, the function $(t, x) \mapsto T(t)f(x)$ is smooth in $(0, +\infty) \times \mathbb{R}^d$, and we have*

$$\frac{d}{dt}(T(t)f)(x) = \Delta T(t)f(x) - x \cdot \nabla T(t)f(x), \quad t > 0,\ x \in \mathbb{R}^d. \tag{13.1.2}$$

If $f\in C^2_b(\mathbb{R}^d)$, for every $x\in\mathbb{R}^d$ the function $t\mapsto T(t)f(x)$ is differentiable also at $t=0$, with

$$\frac{d}{dt}(T(t)f)(x)_{|t=0}=\Delta f(x)-x\cdot\nabla f(x),\quad x\in\mathbb{R}^d, \tag{13.1.3}$$

and the function $(t,x)\mapsto d/dt\,(T(t)f)(x)$ is continuous in $[0,+\infty)\times\mathbb{R}^d$.

Proof. Setting $z=e^{-t}x+\sqrt{1-e^{-2t}}y$ in (12.1.1) we see that $(t,x)\mapsto T(t)f(x)$ is smooth in $(0,+\infty)\times\mathbb{R}^d$, and that

$$\begin{aligned}\frac{d}{dt}(T(t)f)(x)&=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}f(z)\frac{\partial}{\partial t}\bigg(\exp\Big\{-\frac{|z-e^{-t}x|^2}{2(1-e^{-2t})}\Big\}(1-e^{-2t})^{-d/2}\bigg)dz\\
&=\frac{(1-e^{-2t})^{-d/2}}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}f(z)\exp\Big\{-\frac{|z-e^{-t}x|^2}{2(1-e^{-2t})}\Big\}\\
&\qquad\bigg(-\frac{e^{-t}(z-e^{-t}x)\cdot x}{1-e^{-2t}}+\frac{e^{-2t}|z-e^{-t}x|^2}{(1-e^{-2t})^2}-\frac{de^{-2t}}{1-e^{-2t}}\bigg)dz\\
&=\int_{\mathbb{R}^d}f(e^{-t}x+\sqrt{1-e^{-2t}}y)(-c(t)\,y\cdot x+c(t)^2|y|^2-dc(t)^2)\gamma_d(dy),\end{aligned}$$

where $c(t)=e^{-t}/\sqrt{1-e^{-2t}}$. Differentiating twice with respect to x in (12.1.1) (recall (12.1.5)), we obtain

$$D_{ij}(T(t)f)(x)=c(t)^2\int_{\mathbb{R}^d}f(e^{-t}x+\sqrt{1-e^{-2t}}y)(-\delta_{ij}+y_iy_j)\gamma_d(dy)$$

so that

$$\Delta T(t)f(x)=c(t)^2\int_{\mathbb{R}^d}f(e^{-t}x+\sqrt{1-e^{-2t}}y)(-d+|y|^2)\gamma_d(dy).$$

Therefore,

$$\begin{aligned}\frac{d}{dt}(T(t)f)(x)-\Delta T(t)f(x)&=-c(t)\int_{\mathbb{R}^d}f(e^{-t}x+\sqrt{1-e^{-2t}}y)\,x\cdot y\,\gamma_d(dy)\\
&=-\nabla T(t)f(x)\cdot x,\end{aligned}$$

and (13.1.2) follows.

For $f\in C^2_b(\mathbb{R}^d)$, we rewrite formula (13.1.2) as

$$\begin{aligned}\frac{d}{dt}(T(t)f)(x)&=(\mathcal{L}T(t)f)(x)\\
&=e^{-2t}(T(t)\Delta f)(x)-e^{-t}x\cdot(T(t)\nabla f)(x),\quad t>0,\ x\in\mathbb{R}^d,\end{aligned} \tag{13.1.4}$$

taking into account (12.1.7) and (12.1.9). (We recall that $(T(t)\nabla f)(x)$ is the vector whose j-th component is $T(t)D_jf(x)$). Since Δf and each D_jf are continuous and bounded in $\mathbb{R}^d$, the right hand side is continuous in $[0,+\infty)\times\mathbb{R}^d$. So, for every $x\in\mathbb{R}^d$ the function $\theta(t):=T(t)f(x)$ is continuous in $[0,+\infty)$, it is differentiable in $(0,+\infty)$ and $\lim_{t\to0}\theta'(t)=\Delta f(x)-x\cdot\nabla f(x)$. Therefore, θ is differentiable at 0 too, (13.1.4) holds also at $t=0$, and (13.1.3) follows. □

Lemma 13.1.1 suggests that L_p is a suitable realisation of the Ornstein-Uhlenbeck differential operator $\mathcal{L}$ defined in (13.1.1). For a first characterisation of L_p we use Lemma 11.1.9.

Proposition 13.1.2. *For $1 \le p < \infty$ and $k \in \mathbb{N}$, $k \ge 2$, the operator $\mathcal{L} : D(\mathcal{L}) = C_b^k(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, \gamma_d) \to L^p(\mathbb{R}^d, \gamma_d)$ is closable, and its closure is L_p. So, $D(L_p)$ consists of all $f \in L^p(\mathbb{R}^d, \gamma_d)$ for which there exists a sequence of functions $(f_n) \subset C_b^k(\mathbb{R}^d)$ such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $(\mathcal{L}f_n)$ converges in $L^p(\mathbb{R}^d, \gamma_d)$. In this case, $L_p f = L^p - \lim_{n\to\infty} \mathcal{L}f_n$.*

Proof. We check that $D = C_b^k(\mathbb{R}^d)$ satisfies the assumptions of Proposition 11.1.9, i.e., it is a core of L_p. We already know, from Proposition 12.1.4, that $T(t)$ maps $C_b^k(\mathbb{R}^d)$ into itself for $k = 1$, 2. The proof of the fact that $C_b^k(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \gamma_d)$ and that $T(t)$ maps $C_b^k(\mathbb{R}^d)$ into itself for $k \ge 3$ is left as Exercise 13.2.

Since $C_b^k(\mathbb{R}^d) \subset C_b^2(\mathbb{R}^d)$ for $k \ge 2$, it remains to prove that $C_b^2(\mathbb{R}^d) \subset D(L_p)$, and $L_p f = \mathcal{L}f$ for every $f \in C_b^2(\mathbb{R}^d)$.

By Lemma 13.1.1, for every $f \in C_b^2(\mathbb{R}^d)$ we have $d/dt\, T(t)f(x) = \mathcal{L}T(t)f(x)$ for every $x \in \mathbb{R}^d$ and $t \ge 0$; moreover $t \mapsto d/dt\, T(t)f(x)$ is continuous for every x. Therefore for every $t > 0$ we have

$$\frac{T(t)f(x) - f(x)}{t} = \frac{1}{t}\int_0^t \frac{d}{ds}T(s)f(x)\,ds = \frac{1}{t}\int_0^t \mathcal{L}T(s)f(x)\,ds,$$

and

$$\int_{\mathbb{R}^d} \left|\frac{T(t)f(x) - f(x)}{t} - \mathcal{L}f(x)\right|^p \gamma_d(dx) \le \int_{\mathbb{R}^d} \left(\frac{1}{t}\int_0^t |\mathcal{L}T(s)f(x) - \mathcal{L}f(x)|\,ds\right)^p \gamma_d(dx).$$

Since $s \mapsto \mathcal{L}T(s)f(x)$ is continuous, for every x we have

$$\lim_{t\to 0^+} \left(\frac{1}{t}\int_0^t |\mathcal{L}T(s)f(x) - \mathcal{L}f(x)|\,ds\right)^p = 0.$$

Moreover, by (13.1.4),

$$\left(\frac{1}{t}\int_0^t |\mathcal{L}T(s)f(x) - \mathcal{L}f(x)|\,ds\right)^p \le 2^p(\|\Delta f\|_\infty + |x|\,\|\,|\nabla f|\,\|_\infty)^p \in L^1(\mathbb{R}^d, \gamma_d).$$

By the Dominated Convergence Theorem,

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} \left|\frac{T(t)f(x) - f(x)}{t} - \mathcal{L}f(x)\right|^p \gamma_d(dx) = 0.$$

Then, $C_b^k(\mathbb{R}^d) \subset D(L_p)$. Since L_p is a closed operator and it is an extension of $\mathcal{L} : C_b^k(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d)$, $\mathcal{L} : C_b^k(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d)$ is closable.

Applying Lemma 11.1.9 with $D = C_b^k(\mathbb{R}^d)$, we obtain that $D(L_p)$ is the closure of $C_b^k(\mathbb{R}^d)$ in the graph norm of L_p, namely $f \in D(L_p)$ iff there exists a sequence $(f_n) \subset C_b^k(\mathbb{R}^d)$ such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $L_p f_n = \mathcal{L}f_n$ converges in $L^p(\mathbb{R}^d, \gamma_d)$. This shows that L_p is the closure of $\mathcal{L} : C_b^k(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d)$. □

In the case $p = 2$ we obtain other characterisations of $D(L_2)$. To start with, we point out some important properties of $\mathcal{L}$, when applied to elements of $W^{2,2}(\mathbb{R}^d, \gamma_d)$.

Lemma 13.1.3. *(a) $\mathcal{L} : W^{2,2}(\mathbb{R}^d, \gamma_d) \to L^2(\mathbb{R}^d, \gamma_d)$ is a bounded operator;*

(b) for every $f \in W^{2,2}(\mathbb{R}^d, \gamma_d)$, $g \in W^{1,2}(\mathbb{R}^d, \gamma_d)$ we have

$$\int_{\mathbb{R}^d} \mathcal{L}f\, g\, d\gamma_d = -\int_{\mathbb{R}^d} \nabla f \cdot \nabla g\, d\gamma_d. \tag{13.1.5}$$

(c) for every $f \in W^{2,2}(\mathbb{R}^d, \gamma_d)$ we have

$$\mathcal{L}f = \mathrm{div}_{\gamma_d} \nabla f. \tag{13.1.6}$$

Proof. To prove (a) it is sufficient to show that the mapping $T : W^{2,2}(\mathbb{R}^d, \gamma_d) \to L^2(\mathbb{R}^d, \gamma_d)$ defined by $(Tf)(x) := x \cdot \nabla f(x)$ is bounded. For every $i = 1, \ldots, d$, set $g_i(x) = x_i D_i f(x)$. The mapping $f \mapsto g_i$ is bounded from $W^{2,2}(\mathbb{R}^d, \gamma_d)$ to $L^2(\mathbb{R}^d, \gamma_d)$ by Lemma 10.2.6, and summing up the statement follows.

To prove (b) it is sufficient to apply the integration by parts formula (9.1.3) to compute $\int_{\mathbb{R}^d} D_{ii} f\, g\, d\gamma_d$, for every $i = 1, \ldots, d$, and to sum up. In fact, (9.1.3) was stated for C^1_b functions, but it is readily extended to Sobolev functions using Proposition 9.1.5.

Statement (c) follows from Theorem 10.2.7. In this case we have $H = \mathbb{R}^d$, and it is convenient to take the canonical basis of $\mathbb{R}^d$ as a basis for H. So, we have $\hat{h}_i(x) = x_i$ for $i = 1, \ldots, d$ and $\mathrm{div}_{\gamma_d} v(x) = \sum_{i=1}^d D_i v_i - x_i v_i$, for every $v \in W^{1,2}(\mathbb{R}^d, \gamma_d; \mathbb{R}^d)$. Taking $v = \nabla f$, (13.1.6) follows. □

The first characterisation of $D(L_2)$ is the following.

Theorem 13.1.4. *$D(L_2) = W^{2,2}(\mathbb{R}^d, \gamma_d)$, and $L_2 f = \mathcal{L}f$ for every $f \in W^{2,2}(\mathbb{R}^d, \gamma_d)$. Moreover, for every $f \in W^{2,2}(\mathbb{R}^d, \gamma_d)$,*

$$\|f\|_{L^2(\mathbb{R}^d,\gamma_d)} + \|\mathcal{L}f\|_{L^2(\mathbb{R}^d,\gamma_d)} \le \|f\|_{W^{2,2}(\mathbb{R}^d,\gamma_d)} \le \frac{3}{2}(\|f\|_{L^2(\mathbb{R}^d,\gamma_d)} + \|\mathcal{L}f\|_{L^2(\mathbb{R}^d,\gamma_d)}). \tag{13.1.7}$$

Proof. The embedding $W^{2,2}(\mathbb{R}^d, \gamma_d) \subset D(L_2)$ is an easy consequence of Lemma 13.1.3(a). For every $f \in W^{2,2}(\mathbb{R}^d, \gamma_d)$ there is a sequence (f_k) of C^2_b functions such that $f_k \to f$ in $W^{2,2}(\mathbb{R}^d, \gamma_d)$. Then, $\mathcal{L}f_k \to g = \mathcal{L}f$ in $L^2(\mathbb{R}^d, \gamma_d)$. By Proposition 13.1.2, $f \in D(L_2)$ and $L_2 f = \mathcal{L}f$ for every $f \in W^{2,2}(\mathbb{R}^d, \gamma_d)$. However, the embedding constant that comes from Lemma 13.1.3(a) is not clear, and may depend on d. It is better to use (13.1.6) and Theorem 10.2.7, with $v = \nabla f$, that gives the clean estimate

$$\|\mathcal{L}f\|_{L^2(\mathbb{R}^d,\gamma_d)} \le \|\nabla f\|_{W^{1,2}(\mathbb{R}^d,\gamma_d;\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} \sum_{i=1}^d (D_i f)^2\, d\gamma_d\Big)^{1/2} + \Big(\int_{\mathbb{R}^d} \sum_{i,j=1}^d (D_{ij} f)^2\, d\gamma_d\Big)^{1/2}$$

which yields

$$\|f\|_{L^2(\mathbb{R}^d,\gamma_d)} + \|\mathcal{L}f\|_{L^2(\mathbb{R}^d,\gamma_d)} \le \|f\|_{W^{2,2}(\mathbb{R}^d,\gamma_d)}, \quad f \in W^{2,2}(\mathbb{R}^d, \gamma_d).$$

To prove the other embedding we shall show that

$$\|f\|_{W^{2,2}(\mathbb{R}^d,\gamma_d)} \le \frac{3}{2}(\|f\|_{L^2(\mathbb{R}^d,\gamma_d)} + \|\mathcal{L}f\|_{L^2(\mathbb{R}^d,\gamma_d)}) \tag{13.1.8}$$

for every $f \in C_b^3(\mathbb{R}^d)$. Indeed, by Proposition 13.1.2, if $f \in D(L_2)$ there is a sequence (f_k) of C_b^3 functions such that f_k converges to f and $\mathcal{L}f_k$ converges in $L^2(\mathbb{R}^d,\gamma_d)$. Applying estimate (13.1.8) to $f_k - f_h$ we obtain that (f_k) is a Cauchy sequence in $W^{2,2}(\mathbb{R}^d,\gamma_d)$, so that its L^2 limit f belongs to $W^{2,2}(\mathbb{R}^d,\gamma_d)$, and f satisfies estimate (13.1.8), too. This shows that $D(L_2)$ is continuously embedded in $W^{2,2}(\mathbb{R}^d,\gamma_d)$ and that (13.1.8) holds for every $f \in D(L_2)$.

So, let us prove that (13.1.8) holds for every $f \in C_b^3(\mathbb{R}^d)$. By Lemma 13.1.3(b),

$$\int_{\mathbb{R}^d} |\nabla f|^2 d\gamma_d = -\int_{\mathbb{R}^d} f\mathcal{L}f\, d\gamma_d \le \|f\|_{L^2(\mathbb{R}^d,\gamma_d)}\|\mathcal{L}f\|_{L^2(\mathbb{R}^d,\gamma_d)}. \tag{13.1.9}$$

To estimate the L^2 norm of the second order derivatives, we set $\mathcal{L}f =: g$ and we differentiate with respect to x_j (this is why we consider C_b^3, instead of only C_b^2, functions) for every $j = 1, \dots d$. We obtain

$$D_j(\Delta f) - \sum_{i=1}^{d}(\delta_{ij}D_i f + x_i D_{ji} f) = D_j g.$$

Multiplying by $D_j f$ and summing up we get

$$\sum_{j=1}^{d} D_j f \Delta(D_j f) - |\nabla f|^2 - \sum_{j=1}^{d} x \cdot \nabla(D_j f) D_j f = \nabla f \cdot \nabla g.$$

Note that each term in the above sum belongs to $L^p(\mathbb{R}^d,\gamma_d)$ for every $p > 1$. We integrate over $\mathbb{R}^d$ and we obtain

$$\int_{\mathbb{R}^d} \Big(\sum_{j=1}^{d} D_j f\, \mathcal{L}(D_j f) - |\nabla f|^2\Big) d\gamma_d = \int_{\mathbb{R}^d} \nabla f \cdot \nabla g\, d\gamma_d.$$

Now we use the integration formula (13.1.5), both in the left hand side and in the right hand side, obtaining

$$-\int_{\mathbb{R}^d} \sum_{j=1}^{d} |\nabla D_j f|^2 d\gamma_d - \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma_d = -\int_{\mathbb{R}^d} g\, \mathcal{L}f\, d\gamma_d$$

so that, since $g = \mathcal{L}f$

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^{d} (D_{ij} f)^2 d\gamma_d = \int_{\mathbb{R}^d} (\mathcal{L}f)^2 d\gamma_d - \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma_d \le \int_{\mathbb{R}^d} (\mathcal{L}f)^2 d\gamma_d.$$

This inequality and (13.1.9) yield (13.1.8). Indeed,

$$\begin{aligned}\|f\|_{W^{2,2}} &\le \|f\|_{L^2} + (\|f\|_{L^2}\|\mathcal{L}f\|_{L^2})^{1/2} + \|\mathcal{L}f\|_{L^2} \\ &\le \|f\|_{L^2} + \frac{1}{2}(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2}) + \|\mathcal{L}f\|_{L^2} \\ &\le \frac{3}{2}(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2}).\end{aligned}$$

□

The next characterisation fits last year Isem. We recall below general results about bilinear forms in Hilbert spaces. We only need a basic result; more refined results are in last year Isem lecture notes.

Let $V \subset W$ be real Hilbert spaces, with continuous and dense embedding, and let $\mathcal{Q} : V \times V \to \mathbb{R}$ be a bounded bilinear form. "Bounded" means that there exists $M > 0$ such that $|\mathcal{Q}(u, v)| \le M\|u\|_V\, \|v\|_V$ for every u, $v \in V$; "bilinear" means that $\mathcal{Q}$ is linear both with respect to u and with respect to v. $\mathcal{Q}$ is called "nonnegative" if $\mathcal{Q}(u, u) \ge 0$ for every $u \in V$, and "coercive" if there is $c > 0$ such that $\mathcal{Q}(u, u) \ge c\|u\|_V^2$, for every $u \in V$; it is called "symmetric" if $\mathcal{Q}(u, v) = \mathcal{Q}(v, u)$ for every u, $v \in V$. Note that the form in (13.1.10), with $V = W^{1,2}(\mathbb{R}^d, \gamma_d)$, $W = L^2(\mathbb{R}^d, \gamma_d)$ is bounded, bilinear, symmetric and nonnegative. It is not coercive, but $\mathcal{Q}(u, v) + \alpha\langle u, v\rangle_{L^2(\mathbb{R}^d,\gamma_d)}$ is coercive for every $\alpha > 0$.

For any bounded bilinear form $\mathcal{Q}$, an unbounded linear operator A in the space W is naturally associated with $\mathcal{Q}$. $D(A)$ consists of the elements $u \in V$ such that the mapping $V \to \mathbb{R}$, $v \mapsto \mathcal{Q}(u, v)$, has a linear bounded extension to the whole W. By the Riesz Theorem, this is equivalent to the existence of $g \in W$ such that $\mathcal{Q}(u, v) = \langle g, v\rangle_W$, for every $v \in V$. Note that g is unique, because V is dense in W. Then we set $Au = -g$, where g is the unique element of W such that $\mathcal{Q}(u, v) = \langle g, v\rangle_W$, for every $v \in V$.

Theorem 13.1.5. *Let $V \subset W$ be real Hilbert spaces, with continuous and dense embedding, and let $\mathcal{Q} : V \times V \to \mathbb{R}$ be a bounded bilinear symmetric form, such that $(u, v) \mapsto \mathcal{Q}(u, v) + \alpha\langle u, v\rangle_W$ is coercive for some $\alpha > 0$. Then the operator $A : D(A) \to W$ defined above is densely defined and self-adjoint. If in addition $\mathcal{Q}$ is nonnegative, A is dissipative.*

Proof. The mapping $(u, v) \mapsto \mathcal{Q}(u, v) + \alpha\langle u, v\rangle_W$ is an inner product in V, and the associated norm is equivalent to the V-norm, by the continuity of $\mathcal{Q}$ and the coercivity assumption.

It is convenient to consider the operator $\widetilde{A} : D(\widetilde{A}) = D(A) \to W$, $\widetilde{A}u := Au + \alpha u$. Of course if $\widetilde{A}$ is self-adjoint, also A is self-adjoint.

We consider the canonical isomorphism $T : V \to V^*$ defined by $(Tu)(v) = \mathcal{Q}(u, v) + \alpha\langle u, v\rangle_W$ (we are using the new inner product above defined), and the embedding $J : W \to V^*$, such that $(Ju)(v) = \langle u, v\rangle_W$. T is an isometry by the Riesz Theorem, and J is bounded since for every $u \in W$ and $v \in V$ we have $|(Ju)(v)| \le \|u\|_W\|v\|_W \le C\|u\|_W\|v\|_V$, where C is the norm of the embedding $V \subset W$. Moreover, J is one to one, since V is dense in W.

By definition, $u \in D(\widetilde{A})$ iff there exists $g \in W$ such that $\mathcal{Q}(u,v) + \alpha\langle u,v\rangle_W = \langle g,v\rangle_W$ for every $v \in V$, which means $Tu = Jg$, and in this case $\widetilde{A}u = -g$.

The range of J is dense in V^*. If it were not, there would exist $\Phi \in V^* \setminus \{0\}$ such that $\langle Jw, \Phi\rangle_{V^*} = 0$ for every $w \in W$. So, there would exists $\varphi \in V \setminus \{0\}$ such that $Jw(\varphi) = 0$, namely $\langle w, \varphi\rangle_W = 0$ for every $w \in W$. This implies $\varphi = 0$, a contradiction. Since T is an isomorphism, the range of $T^{-1}J$, which is nothing but the domain of $\widetilde{A}$, is dense in V. Since V is in its turn dense in W, then $D(\widetilde{A})$ is dense in W.

The symmetry of $\mathcal{Q}$ implies immediately that $\widetilde{A}$ is self–adjoint. Indeed, for $u, v \in D(\widetilde{A})$ we have

$$\langle \widetilde{A}u, v\rangle_W = \mathcal{Q}(u,v) + \alpha\langle u,v\rangle_W = \mathcal{Q}(v,u) + \alpha\langle v,u\rangle_W = \langle u, \widetilde{A}v\rangle_W.$$

Since $\widetilde{A}$ is onto, it is self-adjoint.

The last statement is obvious: since $\langle Au, u\rangle = -\mathcal{Q}(u,u)$ for every $u \in D(A)$, if $\mathcal{Q}$ is nonnegative then A is dissipative. □

In our setting the bilinear form is

$$\mathcal{Q}(u,v) := \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, d\gamma_d, \quad u,\ v \in W^{1,2}(\mathbb{R}^d, \gamma_d), \tag{13.1.10}$$

so that the assumptions of Theorem 13.1.5 are satisfied with $W = L^2(\mathbb{R}^d, \gamma_d)$, $V = W^{1,2}(\mathbb{R}^d, \gamma_d)$ and every $\alpha > 0$. $D(A)$ is the set

$$\Big\{u \in W^{1,2}(\mathbb{R}^d, \gamma_d) :\ \exists g \in L^2(\mathbb{R}^d, \gamma_d) \text{ such that } \mathcal{Q}(u,v) = \int_{\mathbb{R}^d} g\, v \, d\gamma_d,\ \forall v \in W^{1,2}(\mathbb{R}^d, \gamma_d)\Big\}$$

and $Au = -g$.

Theorem 13.1.6. *Let $\mathcal{Q}$ be the bilinear form in* (13.1.10)*. Then $D(A) = W^{2,2}(\mathbb{R}^d, \gamma_d)$, and $A = L_2$.*

Proof. Let $u \in W^{2,2}(\mathbb{R}^d, \gamma_d)$. By (13.1.5) and Theorem 13.1.4, for every $v \in W^{1,2}(\mathbb{R}^d, \gamma_d)$ we have

$$\mathcal{Q}(u,v) = -\int_{\mathbb{R}^d} \mathcal{L}u\, v \, d\gamma_d$$

Therefore, the function $g = \mathcal{L}u = L_2 u$ fits the definition of Au (recall that $g \in L^2(\mathbb{R}^d, \gamma_d)$ by Lemma 13.1.3(a)). So, $W^{2,2}(\mathbb{R}^d, \gamma_d) \subset D(A)$ and $Au = L_2 u$ for every $u \in W^{2,2}(\mathbb{R}^d, \gamma_d)$ (the last equality follows from Theorem 13.1.4). In other words, A is a self-adjoint extension of L_2. L_2 itself is self-adjoint by Corollary 11.4.5, because $T_2(t)$ is self-adjoint in $L^2(\mathbb{R}^d, \gamma_d)$ by Proposition 12.1.5(ii), for every $t > 0$. Self-adjoint operators have no proper self-adjoint extensions (this is because $D(L_2) \subset D(A) \Rightarrow D(A^*) \subset D(L_2^*)$, but $D(A^*) = D(A)$ and $D(L_2^*) = D(L_2)$), hence $D(A) = D(L_2)$ and $A = L_2$. □

13.2 The infinite dimensional case

Here, as usual, X is a separable Banach space endowed with a centred nondegenerate Gaussian measure γ, and H is the relevant Cameron-Martin space.

The connection between finite dimension and infinite dimension is provided by the cylindrical functions. In the next proposition we show that suitable cylindrical functions belong to $D(L_p)$ for every $p \in (1, +\infty)$, and we write down an explicit expression of $L_p f$ for such f. Precisely, we fix an orthonormal basis $\{h_j : j \in \mathbb{N}\}$ of H contained in $R_\gamma(X^*)$, and we denote by Σ the set of the cylindrical functions of the type $f(x) = \varphi(\hat{h}_1(x), \ldots, \hat{h}_d(x))$ with $\varphi \in C_b^2(\mathbb{R}^d)$, for some $d \in \mathbb{N}$. This is a dense subspace of $L^p(X, \gamma)$ for every $p \in [1, +\infty)$, see Exercise 13.3. For such f, we have

$$\partial_i f(x) = \frac{\partial \varphi}{\partial \xi_i}(\hat{h}_1(x), \ldots, \hat{h}_d(x)),\ i \le d; \quad \partial_i f(x) = 0,\ i > d. \tag{13.2.1}$$

To distinguish between the finite and the infinite dimensional case, we use the superscript (d) when dealing with the Ornstein-Uhlenbeck semigroup and the Ornstein-Uhlenbeck semigroup in $\mathbb{R}^d$. So, $L_p^{(d)}$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $T^{(d)}(t)$ in $L^p(\mathbb{R}^d, \gamma_d)$. We recall that $L_p^{(d)}$ is a realisation of the operator $\mathcal{L}^{(d)} = \Delta - x \cdot \nabla$, namely $L_p^{(d)} f = \mathcal{L}^{(d)} f$ for every $f \in D(L_p^{(d)})$.

Proposition 13.2.1. *Let $\{h_j : j \in \mathbb{N}\}$ be any orthonormal basis of H contained in $R_\gamma(X^*)$, and let $f(x) = \varphi(\hat{h}_1(x), \ldots, \hat{h}_d(x))$ with $\varphi \in L^p(\mathbb{R}^d, \gamma_d)$, for some $d \in \mathbb{N}$ and $p \in [1, +\infty)$. Then for every $t > 0$ and γ-a.e. $x \in X$,*

$$T_p(t) f(x) = (T_p^{(d)}(t)\varphi)(\hat{h}_1(x), \ldots, \hat{h}_d(x)).$$

If in addition $\varphi \in D(L_p^{(d)})$, then $f \in D(L_p)$, and

$$L_p f(x) = L_p^{(d)} \varphi(\hat{h}_1(x), \ldots, \hat{h}_d(x)).$$

If $\varphi \in C_b^2(\mathbb{R}^d)$, then

$$L_p f(x) = \mathcal{L}^{(d)} \varphi(\hat{h}_1(x), \ldots, \hat{h}_d(x)) = \sum_{i=1}^{d} (\partial_{ii} f(x) - \hat{h}_i(x)\partial_i f(x)) = \mathrm{div}_\gamma \nabla_H f(x).$$

Proof. Assume first that $\varphi \in C_b(\mathbb{R}^d)$. For $t > 0$ we have

$$\begin{aligned}
T(t) f(x) &= \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\gamma(dy) \\
&= \int_X \varphi(e^{-t}\hat{h}_1(x) + \sqrt{1 - e^{-2t}}\hat{h}_1(y), \ldots, e^{-t}\hat{h}_d(x) + \sqrt{1 - e^{-2t}}\hat{h}_d(y))\gamma(dy) \\
&= \int_{\mathbb{R}^d} \varphi(e^{-t}\hat{h}_1(x) + \sqrt{1 - e^{-2t}}\xi_1, \ldots, e^{-t}\hat{h}_d(x) + \sqrt{1 - e^{-2t}}\xi_d)\gamma_d(d\xi) \\
&= (T^{(d)}(t)\varphi)(\hat{h}_1(x), \ldots, \hat{h}_d(x)),
\end{aligned}$$

because $\gamma \circ (\hat{h}_1, \dots, \hat{h}_d)^{-1} = \gamma_d$ by Exercise 2.4. If $\varphi \in L^p(\mathbb{R}^d, \gamma_d)$ is not continuous, we approximate it in $L^p(\mathbb{R}^d, \gamma_d)$ by a sequence of continuous and bounded functions φ_n. The sequence $f_n(x) := \varphi_n(\hat{h}_1(x), \dots, \hat{h}_d(x))$ converges to f and the sequence $g_n(x) := (T^{(d)}(t)\varphi_n)(\hat{h}_1(x), \dots, \hat{h}_d(x))$ converges to $(T^{(d)}(t)\varphi)(\hat{h}_1(x), \dots, \hat{h}_d(x))$ in $L^p(X, \gamma)$, still by Exercise 2.4. Therefore, $T(t)f_n$ converges to $T(t)f$ in $L^p(X, \gamma)$ for every $t > 0$, and the first statement follows.

Let now $\varphi \in D(L_p^{(d)})$. For every $t > 0$ we have

$$\begin{aligned}
&\int_X \Big| \frac{T(t)f(x) - f(x)}{t} - L_p^{(d)}\varphi(\hat{h}_1(x), \dots, \hat{h}_d(x)) \Big|^p \gamma(dx) \\
&= \int_X \Big| \frac{T^{(d)}(t)\varphi(\hat{h}_1(x), \dots, \hat{h}_d(x)) - \varphi(\hat{h}_1(x), \dots, \hat{h}_d(x))}{t} - L_p^{(d)}\varphi(\hat{h}_1(x), \dots, \hat{h}_d(x)) \Big|^p \gamma(dx) \\
&= \int_{\mathbb{R}^d} \Big| \frac{T^{(d)}(t)\varphi(\xi) - \varphi(\xi)}{t} - L_p^{(d)}\varphi(\xi) \Big|^p \gamma_d(d\xi)
\end{aligned}$$

that vanishes as $t \to 0$. So, the second statement follows.

Let $\varphi \in C_b^2(\mathbb{R}^d)$. By Theorem 13.1.2 we have

$$L_p^{(d)}\varphi(\xi) = \sum_{i=1}^d (D_{ii}\varphi(\xi) - \xi_i D_i\varphi(\xi)) = \mathcal{L}^{(d)}\varphi(\xi), \quad \xi \in \mathbb{R}^d.$$

Therefore,

$$\begin{aligned}
L_p f(x) &= (\mathcal{L}^{(d)}\varphi)(\hat{h}_1(x), \dots, \hat{h}_d(x)) = \sum_{i=1}^d (D_{ii}\varphi(\xi) - \xi_i D_i\varphi(\xi))_{|\xi=(\hat{h}_1(x),\dots,\hat{h}_d(x))} \\
&= \sum_{i=1}^d (\partial_{ii} f(x) - \hat{h}_i(x)\partial_i f(x)),
\end{aligned}$$

which coincides with $\text{div}_\gamma \nabla_H f(x)$. See Theorem 10.2.7. □

As a consequence of Propositions 13.2.1 and 11.1.9, we obtain a characterisation of $D(L_p)$ which is quite similar to the finite dimensional one.

Theorem 13.2.2. *Let $\{h_j : j \in \mathbb{N}\}$ be an orthonormal basis of H contained in $R_\gamma(X^*)$. Then the subspace Σ of $\mathcal{F}C_b^2(X)$ defined above is a core of L_p for every $p \in [1, +\infty)$, the restriction of L_p to Σ is closable in $L^p(X, \gamma)$ and its closure is L_p. In other words, $D(L_p)$ consists of all $f \in L^p(X, \gamma)$ such that there exists a sequence (f_n) in Σ which converges to f in $L^p(X, \gamma)$ and such that $L_p f_n = \text{div}_\gamma \nabla_H f_n$ converges in $L^p(X, \gamma)$.*

Proof. By Proposition 13.2.1, $\Sigma \subset D(L_p)$. For every $t > 0$, $T(t)f \in \Sigma$ if $f \in \Sigma$, by Proposition 13.2.1 and Proposition 12.1.4. By Lemma 11.1.9, Σ is a core of L_p. □

For $p = 2$ we can prove other characterisations.

Theorem 13.2.3. *$D(L_2) = W^{2,2}(X,\gamma)$, and for every $f \in W^{2,2}(X,\gamma)$ we have*

$$L_2 f = \mathrm{div}_\gamma \nabla_H f, \tag{13.2.2}$$

and

$$\|f\|_{L^2(X,\gamma)} + \|L_2 f\|_{L^2(X,\gamma)} \le \|f\|_{W^{2,2}(X,\gamma)} \le \frac{3}{2}(\|f\|_{L^2(X,\gamma)} + \|L_2 f\|_{L^2(X,\gamma)}). \tag{13.2.3}$$

Proof. Fix an orthonormal basis of H contained in $R_\gamma(X^*)$. By Exercise 13.3, Σ is dense in $W^{2,2}(X,\gamma)$, and by Theorem 13.2.2 it is dense in $D(L_2)$.

We claim that every $f \in \Sigma$ satisfies (13.2.3), so that the $W^{2,2}$ norm is equivalent to the graph norm of L_2 on Σ. For every $f \in \Sigma$, if $f(x) = \varphi(\hat{h}_1(x), \ldots, \hat{h}_d(x))$, by Proposition 13.2.1 we have $L_2 f(x) = (\mathcal{L}^{(d)}\varphi)(\hat{h}_1(x), \ldots, \hat{h}_d(x))$, where $\mathcal{L}^{(d)}$ is defined in (13.1.1). Recalling that $\gamma \circ (\hat{h}_1, \ldots, \hat{h}_d)^{-1} = \gamma_d$, we get

$$\int_X f^2 d\gamma = \int_{\mathbb{R}^d} \varphi^2 d\gamma_d, \quad \int_X (L_2 f)^2 d\gamma = \int_{\mathbb{R}^d} (\mathcal{L}^{(d)}\varphi)^2 d\gamma_d,$$

and, using (13.2.1),

$$\|f\|_{W^{2,2}(X,\gamma)} = \|\varphi\|_{W^{2,2}(\mathbb{R}^d,\gamma_d)}.$$

Therefore, estimates (13.1.7) imply that f satisfies (13.2.3), and the claim is proved.

The statement is now a standard consequence of the density of Σ in $W^{2,2}(X,\gamma)$ and in $D(L_2)$. Indeed, to prove that $W^{2,2}(X,\gamma) \subset D(L_2)$, and that $L_2 f = \mathrm{div}_\gamma \nabla_H f$ for every $f \in W^{2,2}(\mathbb{R}^d,\gamma_d)$, it is sufficient to approximate any $f \in W^{2,2}(X,\gamma)$ by a sequence (f_n) of elements of Σ; then f_n converges to f and $L_2 f_n = \mathrm{div}_\gamma \nabla_H f_n$ converges to $\mathrm{div}_\gamma \nabla_H f$ in $L^2(X,\gamma)$ by Theorem 10.2.7, as $\nabla_H f_n$ converges to $\nabla_H f$ in $L^2(X,\gamma;H)$. Since L_2 is a closed operator, $f \in D(L_2)$ and $L_2 f = \mathrm{div}_\gamma \nabla_H f$. Similarly, to prove that $D(L_2) \subset W^{2,2}(X,\gamma)$ we approximate any $f \in D(L_2)$ by a sequence (f_n) of elements of Σ that converges to f in the graph norm; then (f_n) is a Cauchy sequence in $W^{2,2}(X,\gamma)$ and therefore $f \in W^{2,2}(X,\gamma)$. □

Eventually, as in finite dimension, we have a characterisation of L_2 in terms of the bilinear form

$$\mathcal{Q}(u,v) = \int_X [\nabla_H u, \nabla_H v]_H d\gamma, \quad u,\ v \in W^{1,2}(X,\gamma). \tag{13.2.4}$$

Applying Theorem 13.1.5 with $W = L^2(X,\gamma)$, $V = W^{1,2}(X,\gamma)$ we obtain

Theorem 13.2.4. *Let A be the operator associated with the bilinear form $\mathcal{Q}$ above. Then $D(A) = W^{2,2}(X,\gamma)$, and $A = L_2$.*

The proof is identical to the proof of Theorem 13.1.6 and it is omitted.

Note that Theorem 13.2.4 implies that for every $f \in D(L_2) = W^{2,2}(X,\gamma)$ and for every $g \in W^{1,2}(X,\gamma)$ we have

$$\int_X L_2 f\, g\, d\gamma = -\int_X [\nabla_H f, \nabla_H g]_H d\gamma \tag{13.2.5}$$

that is the infinite dimensional version of (13.1.5). Proposition 11.4.3 implies that L_2 is a sectorial operator, therefore the Ornstein-Uhlenbeck semigroup is analytic in $L^2(X,\gamma)$.

We mention that, by general results about semigroups and interpolation theory (e.g. [D, Thm. 1.4.2]), $\{T_p(t):\ p \geq 0\}$ is an analytic semigroup in $L^p(X,\gamma)$ for every $p \in (1,+\infty)$. However, this fact will not be used in these lectures.

A result similar to Theorem 13.2.3 holds also for $p \neq 2$. More precisely, for every $p \in (1,+\infty)$, $D(L_p) = W^{2,p}(X,\gamma)$, and the graph norm of $D(L_p)$ is equivalent to the $W^{2,p}$ norm. But the proof is not as simple. We refer to [M] and [B, Sect. 5.5] for the infinite dimensional case, and to [MPRS] for an alternative proof in the finite dimensional case.

13.3 Exercises

Exercise 13.1. Let $\varrho : \mathbb{R}^d \to [0,+\infty)$ be a mollifier, i.e. a smooth function with support in $B(0,1)$ such that

$$\int_{B(0,1)} \varrho(x)dx = 1.$$

For $\varepsilon > 0$ set

$$\varrho_\varepsilon(x) = \varepsilon^{-d}\varrho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d.$$

Prove that if $p \in [1,+\infty)$ and $f \in L^p(\mathbb{R}^d,\gamma_d)$, then

$$f_\varepsilon(x) := f * \varrho_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\varrho_\varepsilon(x-y)dy,$$

is well defined, belongs to $L^p(\mathbb{R}^d,\gamma_d)$ and converges to f in $L^p(\mathbb{R}^d,\gamma_d)$ as $\varepsilon \to 0^+$.

Exercise 13.2. Prove that for every $k \in \mathbb{N}$, $k \geq 3$, $C^k_b(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d,\gamma_d)$ and that $T(t) \in \mathcal{L}(C^k_b(\mathbb{R}^d))$ for every $t > 0$.

Exercise 13.3. Let $\{h_j :\ j \in \mathbb{N}\}$ be any orthonormal basis of H contained in $R_\gamma(X^*)$. Prove that the set Σ of the cylindrical functions of the type $f(x) = \varphi(\hat{h}_1(x),\dots,\hat{h}_d(x))$ with $\varphi \in C^2_b(\mathbb{R}^d)$, for some $d \in \mathbb{N}$, is dense in $L^p(X,\gamma)$ and in $W^{2,p}(X,\gamma)$ for every $p \in [1,+\infty)$.

Exercise 13.4.

(i) With the help of Proposition 10.1.2, show that if $f \in W^{1,p}(X,\gamma)$ with $p \in [1,+\infty)$ is such that $\nabla_H f = 0$ a.e., then f is a.e. constant.

(ii) Use point (i) to show that for every $p \in [1,+\infty)$ the kernel of L_p consists of the constant functions.
(Hint: First of all, prove that $T(t)f = f$ for all $f \in D(L_p)$ such that $L_p f = 0$ and then pass to the limit as $t \to +\infty$ in (12.1.3))

Bibliography

[B] V. I. BOGACHEV: *Gaussian Measures.* American Mathematical Society, 1998.

[D] B. DAVIES: *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[M] P.A. MEYER: *Transformations de Riesz pour les lois gaussiennes*, Lect. Notes in Math. **1059** (1984), 179-193.

[MPRS] G. METAFUNE, J. PRÜSS, A. RHANDI, R. SCHNAUBELT: *The domain of the Ornstein-Uhlenbeck operator on a L^p space with invariant measure*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) **1** (2002), 471-485.