Solution to the Exercises of Lecture 13

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Let us start proving a preliminary result concerning shifted essentially bounded functions.

Lemma. The shift operator is continuous with respect to $\|\cdot\|_{L^p(\mathbb{R}^d,\gamma_d)}$ over $L^\infty(\mathbb{R}^d)$ functions, i.e. for every $f \in L^\infty(\mathbb{R}^d)$, it holds $f \in L^p(\mathbb{R}^d, \gamma_d)$ and

$$\lim_{|h|\to 0} \|f(\cdot + h) - f\|_{L^p(\mathbb{R}^d,\gamma_d)} = 0.$$

Proof. Let us fix $p \in [1, +\infty)$, $f \in L^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$. Since γ_d is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$, we have the following continuous immersions:

$$L^\infty(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d, \gamma_d) \hookrightarrow L^p(\mathbb{R}^d, \gamma_d).$$

So we obtain $f \in L^p(\mathbb{R}^d, \gamma_d)$. For all $h \in \mathbb{R}^d$, let us define $f^h := f(\cdot + h)$; obviously $f^h \in L^\infty(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, \gamma_d)$, too. Let us observe that, for all $r > 0$,

$$\int_{B(0,r)^C} |f(x + h) - f(x)|^p \, \gamma_d(dx) \leq (2\|f\|_\infty)^p \, \gamma_d(B(0, r)^C) \quad \forall h \in \mathbb{R}^d,$$

then

$$\lim_{r\to 0^+} \int_{B(0,r)^C} |f(x + h) - f(x)|^p \, \gamma_d(dx) = 0$$

uniformly for $h \in \mathbb{R}^d$. This implies that there exists $r_1 > 0$ such that

$$\int_{B(0,r_1)^C} |f(x + h) - f(x)|^p \, \gamma_d(dx) < \frac{\varepsilon^p}{2} \quad \forall h \in \mathbb{R}^d.$$

Setting G_d the density function of γ_d with respect to the Lebesgue measure $\mathcal{L}^d$, we have $G_d(x) \leq 1$ for every $x \in \mathbb{R}^d$, then

$$\int_{B(0,r_1)} |f(x + h) - f(x)|^p \, \gamma_d(dx) \leq \int_{B(0,r_1)} |f(x + h) - f(x)|^p \, dx.$$

Now let us fix $\delta > 0$ and consider $|h| < \delta$. Since $f \in L^\infty(\mathbb{R}^d) \subset L^p_{loc}(\mathbb{R}^d)$, we have that $f \in L^p(B(0, r_1+\delta))$; it follows that there exists $g \in C_c^\infty(B(0, r_1+\delta))$ such that

$$\|f - g\|_{L^p(B(0,r_1+\delta))} < \frac{\varepsilon}{3 \cdot 2^{\frac{1}{p}}}. \tag{1}$$

By the Triangle Inequality, using the above notation to denote shifted functions, we have

$$\|f^h - f\|_{L^p(B(0,r_1))} \leq \underbrace{\|f^h - g^h\|_{L^p(B(0,r_1))}}_{(I)} + \underbrace{\|g^h - g\|_{L^p(B(0,r_1))}}_{(II)} + \\ + \underbrace{\|g - f\|_{L^p(B(0,r_1))}}_{(III)}. \tag{2}$$

Let us estimate addendum (I) in (2). Since $B(h, r_1) \subset B(0, r_1 + \delta)$, we have

$$\int_{B(0,r_1)} |f(x+h) - g(x+h)|^p \, dx \overset{\substack{z=x+h \\ \downarrow}}{=} \int_{B(h,r_1)} |f(z) - g(z)|^p \, dz \leq \\ \leq \int_{B(0,r_1+\delta)} |f(z) - g(z)|^p \, dz < \frac{1}{2}\left(\frac{\varepsilon}{3}\right)^p,$$

where last inequality follows from (1). So we obtain

$$\|f^h - g^h\|_{L^p(B(0,r_1))} < \frac{\varepsilon}{3 \cdot 2^{\frac{1}{p}}}. \tag{3}$$

To estimate addendum (II), let us observe that g is uniformly continuous on $B(0, r_1 + \delta)$; in particular, there exists $\overline{\delta} \leq \delta$ such that

$$|g(x+h) - g(x)| < \frac{\varepsilon}{3\left[2\mathcal{L}^d(B(0,r_1))\right]^{\frac{1}{p}}}$$

for all $x \in B(0, r_1)$ and for all $|h| < \overline{\delta}$. Then we have

$$\int_{B(0,r_1)} |g(x+h) - g(x)|^p \, dx < \int_{B(0,r_1)} \frac{\varepsilon^p}{3^p 2\mathcal{L}^d(B(0,r_1))} \, dx = \frac{1}{2}\left(\frac{\varepsilon}{3}\right)^p,$$

that is

$$\|g^h - g\|_{L^p(B(0,r_1))} < \frac{\varepsilon}{3 \cdot 2^{\frac{1}{p}}}. \tag{4}$$

Finally, we estimate addendum (III) using (1):

$$\|f - g\|_{L^p(B(0,r_1))} \leq \|f - g\|_{L^p(B(0,r_1+\delta))} < \frac{\varepsilon}{3 \cdot 2^{\frac{1}{p}}}. \tag{5}$$

Using estimates (3), (4), (5) in (2) we have

$$\|f^h - f\|_{L^p(B(0,r_1),\gamma_d)} \leq \|f^h - f\|_{L^p(B(0,r_1))} < 3 \cdot \frac{\varepsilon}{3 \cdot 2^{\frac{1}{p}}} = \frac{\varepsilon}{2^{\frac{1}{p}}}.$$

Then, for every $|h| < \overline{\delta}$, we obtain

$$\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p \, \gamma_d(dx) =$$
$$= \int_{B(0,r_1)} |f(x+h) - f(x)|^p \, \gamma_d(dx) + \int_{B(0,r_1)^C} |f(x+h) - f(x)|^p \, \gamma_d(dx) <$$
$$< \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p,$$

and so

$$\|f^h - f\|_{L^p(\mathbb{R}^d,\gamma_d)} < \varepsilon,$$

proving the thesis. □

Exercise 13.1. Let $\varrho : \mathbb{R}^d \to [0, +\infty)$ be a mollifier, i.e. a smooth function with support in $B(0,1)$ such that

$$\int_{B(0,1)} \varrho(x) \, dx = 1.$$

For $\varepsilon > 0$ set

$$\varrho_\varepsilon(x) := \varepsilon^{-d} \varrho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d.$$

Prove that if $p \in [1, +\infty)$ and $f \in L^\infty(\mathbb{R}^d)$, then

$$f_\varepsilon(x) := f * \varrho_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\varrho_\varepsilon(x-y) \, dy,$$

is well defined, belongs to $L^p(\mathbb{R}^d, \gamma_d)$ and converges to f in $L^p(\mathbb{R}^d, \gamma_d)$ as $\varepsilon \to 0^+$.

Solution. Let us fix $p \in [1, +\infty)$ and prove that the spaces $L^p_{loc}(\mathbb{R}^d, \gamma_d)$ and $L^p_{loc}(\mathbb{R}^d)$ are the same, i.e. that for every $\Omega \subset \mathbb{R}^d$ such that $\overline{\Omega}$ is compact, one has

$$L^p(\Omega, \gamma_d) = L^p(\Omega). \tag{6}$$

Last equality must be intended in a topological sense, not merely setwise.

Let us fix $\Omega \subset \mathbb{R}^d$ with $\overline{\Omega}$ compact and observe that

$$a := \inf_{x \in \Omega} \left[\frac{1}{(2\pi)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{2}\right) \right] > 0.$$

This implies

$$a \int_\Omega |f(x)|^p \, dx \le \int_\Omega |f(x)|^p \, \gamma_d(dx) \le \int_\Omega |f(x)|^p \, dx,$$

which is equivalent to

$$a^{\frac{1}{p}}\|f\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega,\gamma_d)} \le \|f\|_{L^p(\Omega)}.$$

So the norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(\Omega,\gamma_d)}$ are equivalent, then (6) is proved. Now we are ready to prove the required assertion.

Let $f \in L^\infty(\mathbb{R}^d)$. By previous Lemma, $f \in L^p(\mathbb{R}^d,\gamma_d)$ and, by Young's inequality, f_ε is well defined, $f_\varepsilon \in L^\infty(\mathbb{R}^d) \subset L^p(\mathbb{R}^d,\gamma_d)$, with

$$\|f_\varepsilon\|_{L^p(\mathbb{R}^d,\gamma_d)} \le \|f_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le \|f\|_{L^\infty(\mathbb{R}^d)}\|\varrho\|_{L^1(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)}.$$

Now, for $x \in \mathbb{R}^d$, we have

$$f*\varrho_\varepsilon(x)-f(x) = \int_{\mathbb{R}^d}[f(x-y)-f(x)]\varrho_\varepsilon(y)dy \overset{\overset{z=\frac{y}{\varepsilon}}{\downarrow}}{=} \int_{B(0,1)}[f(x-\varepsilon z)-f(x)]\varrho(z)dz.$$

Using Jensen's inequality one obtains

$$|f*\varrho_\varepsilon(x)-f(x)|^p \le \int_{B(0,1)}|f(x-\varepsilon z)-f(x)|^p\varrho(z)\,dz.$$

Integrating last inequality and applying Fubini's Theorem we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}|f*\varrho_\varepsilon(x)-f(x)|^p\,\gamma_d(dx) \le \\
&\quad\le \int_{\mathbb{R}^d}\left(\int_{B(0,1)}|f(x-\varepsilon z)-f(x)|^p\varrho(z)\,dz\right)\gamma_d(dx) = \\
&\quad= \int_{B(0,1)}\varrho(z)\left(\int_{\mathbb{R}^d}|f(x-\varepsilon z)-f(x)|^p\,\gamma_d(dx)\right)dz.
\end{aligned} \tag{7}$$

We observe that the function

$$z \mapsto \int_{\mathbb{R}^d}|f(x-\varepsilon z)-f(x)|^p\,\gamma_d(dx) = \|f(\cdot-\varepsilon z)-f\|^p_{L^p(\mathbb{R}^d,\gamma_d)}$$

converges pointwise to 0 as ε goes to 0^+, by the previous Lemma.

Moreover, since $\|f(\cdot-\varepsilon z)-f\|_{L^p(\mathbb{R}^d,\gamma_d)} \le 2\|f\|_{L^\infty(\mathbb{R}^d)}$, another application of the Dominated Convergence Theorem in formula (7) yields

$$\int_{\mathbb{R}^d}|f*\varrho_\varepsilon(x)-f(x)|^p\,\gamma_d(dx) \overset{\varepsilon\to 0^+}{\longrightarrow} 0,$$

which proves the assertion. □

Exercise 13.2. Prove that for every $k \in \mathbb{N}$, $k \geq 3$, $C_b^k(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \gamma_d)$ and that $T(t) \in \mathcal{L}(C_b^k(\mathbb{R}^d))$ for every $t > 0$.

Solution. For the first assertion, for every fixed $f \in L^p(\mathbb{R}^d, \gamma_d)$ and $\varepsilon > 0$, we have to find a $\overline{f} \in C_b^k(\mathbb{R}^d)$ such that $\|f - \overline{f}\|_{L^p(\mathbb{R}^d,\gamma_d)} < \varepsilon$. Since $f \in L^p(\mathbb{R}^d, \gamma_d)$, we can choose $r > 0$ such that for $\Omega := B(0, r)$ we have

$$\int_{\Omega^c} |f(x)|^p \,\gamma_d(dx) < \left(\frac{\varepsilon}{3}\right)^p,$$

that is

$$\|f - f\mathbb{1}_\Omega\|_{L^p(\mathbb{R}^d,\gamma_d)} < \frac{\varepsilon}{3}. \tag{8}$$

Recalling that $L^p(\Omega, \gamma_d) = L^p(\Omega)$ and $C_c(\Omega)$ is dense in $L^p(\Omega)$, we can find $g \in C_c(\Omega) \subset L^\infty(\Omega)$ such that

$$\|f\mathbb{1}_\Omega - g\|_{L^p(\mathbb{R}^d,\gamma_d)} < \frac{\varepsilon}{3}. \tag{9}$$

Using Exercise 13.1, we can choose $\varepsilon_0 > 0$ such that, for $\overline{f} := g * \varrho_{\varepsilon_0}$, we get

$$\|g - \overline{f}\|_{L^p(\mathbb{R}^d,\gamma_d)} < \frac{\varepsilon}{3}. \tag{10}$$

Inequalities (8), (9), (10) yield

$$\|f - \overline{f}\|_{L^p(\mathbb{R}^d,\gamma_d)} \leq \|f - f\mathbb{1}_\Omega\|_{L^p(\mathbb{R}^d,\gamma_d)} + \|f\mathbb{1}_\Omega - g\|_{L^p(\mathbb{R}^d,\gamma_d)} + \|g - \overline{f}\|_{L^p(\mathbb{R}^d,\gamma_d)} < \varepsilon.$$

Noticing $\overline{f} \in C_b^\infty(\mathbb{R}^d) \subset C_b^k(\mathbb{R}^d)$, with $D^\alpha \overline{f} = g * D^\alpha \varrho_{\varepsilon_0}$ and $\|D^\alpha \overline{f}\|_\infty \leq \|g\|_\infty \|D^\alpha \varrho_{\varepsilon_0}\|_{L^1(\mathbb{R}^d)}$ for every multiindex $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$, we obtain the first assertion.

To prove the second statement, let us fix $k \in \mathbb{N}$, $k \geq 3$, and $f \in C_b^k(\mathbb{R}^d)$. Recalling the expression

$$T(t)f(x) = \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \,\gamma_d(dy),$$

by an iterated use of Proposition 12.14 (or directly by Lebesgue Theorem of differentiation under the integral sign) we get for each $t > 0$ that $T(t)f \in C^k(\mathbb{R}^d)$ and, for every multiindex $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$:

$$\begin{aligned} D^\alpha T(t)f(x) &= e^{-|\alpha|t} \int_{\mathbb{R}^d} D^\alpha f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \,\gamma_d(dy) = \\ &= e^{-|\alpha|t} T(t) D^\alpha f(x). \end{aligned}$$

Last equality combined with Proposition 12.11 yields

$$\|D^\alpha T(t)f\|_\infty = e^{-|\alpha|t}\|T(t)D^\alpha f\|_\infty \le \|D^\alpha f\|_\infty.$$

It follows that

$$\|T(t)f\|_{C_b^k(\mathbb{R}^d)} = \sum_{|\alpha|\le k}\|D^\alpha T(t)f\|_\infty \le \sum_{|\alpha|\le k}\|D^\alpha f\|_\infty = \|f\|_{C_b^k(\mathbb{R}^d)};$$

so $T(t)$ is a contraction over the space $C_b^k(\mathbb{R}^d)$ for every $t > 0$ and the second assertion is proved. □

Exercise 13.3. Let $\{h_j : j \in \mathbb{N}\}$ be an orthonormal basis of H contained in $R_\gamma(X^*)$. Prove that the set Σ of the cylindrical functions of the type $f(x) = \varphi(\hat{h}_1(x), \dots, \hat{h}_d(x))$ with $\varphi \in C_b^2(\mathbb{R}^d)$, for some $d \in \mathbb{N}$, is dense in $L^p(X, \gamma)$ and in $W^{2,p}(X, \gamma)$ for every $p \in [1, +\infty)$.

Solution. Let us prove that Σ is dense in $L^p(\mathbb{R}^d, \gamma_d)$.

Let us fix $f \in L^p(X, \gamma)$ and $\varepsilon > 0$; recalling that $\mathbb{E}_n f \to f$ in $L^p(X, \gamma)$ as n goes to $+\infty$ (Proposition 7.4.5), we can find $d \in \mathbb{N}$ such that

$$\|f - \mathbb{E}_d f\|_{L^p(X,\gamma)} < \frac{\varepsilon}{2}. \tag{11}$$

From Proposition 7.4.1 we have

$$(\mathbb{E}_d f)(x) = \int_X f(P_d x + (I - P_d)y)\,\gamma(dy) = \varphi_d(\hat{h}_1(x), \dots, \hat{h}_d(x)),$$

where

$$\varphi_d(\xi) := \int_X f(\sum_{j=1}^d \xi_j h_j + (I - P_d)y)\,\gamma(dy) \in L^p(\mathbb{R}^d, \gamma_d)$$

since $\gamma_d = \gamma \circ (\hat{h}_1, \dots, \hat{h}_d)^{-1}$. Applying Exercise 13.2, we can find $\psi_d \in C_b^k(\mathbb{R}^d, \gamma_d)$, $k \ge 3$, such that

$$\|\varphi_d - \psi_d\|_{L^p(\mathbb{R}^d,\gamma_d)} < \frac{\varepsilon}{2}.$$

Defining the function g by

$$g(x) := \psi_d(\hat{h}_1(x), \dots, \hat{h}_d(x)),$$

we have $g \in \Sigma$ and, from last inequality, recalling $\gamma_d = \gamma \circ (\hat{h}_1, \dots, \hat{h}_d)^{-1}$,

$$\|g - \mathbb{E}_d f\|_{L^p(X,\gamma)} < \frac{\varepsilon}{2}. \tag{12}$$

By estimates (11) and (12) we obtain

$$\|f-g\|_{L^p(X,\gamma)}<\varepsilon,$$

which proves the first statement.

For the second request, let us notice that the set $\mathcal{F}C_b^2(X)$ is dense in $W^{2,p}(X,\gamma)$ by definition, therefore we only have to prove that Σ is dense in $\mathcal{F}C_b^2(X)$.

To this aim, let us fix $f\in\mathcal{F}C_b^2(X)$ such that $f(x)=\varphi(l_1(x),\dots,l_d(x))$, with $\varphi\in C_b^2(\mathbb{R}^d)$, $l_1,\dots,l_d\in X^*$, $d\in\mathbb{N}$. We will find a sequence $(f_k)_{k\in\mathbb{N}}\subset\Sigma$ such that $f_k\to f$ in $W^{2,p}(X,\gamma)$. Recalling Theorem 7.1.2, we have

$$x=\sum_{n=1}^{+\infty}\hat{h}_n(x)h_n$$

for a.e. $x\in X$; so, for every $j=1,\dots,d$, we have

$$l_j(x)=\sum_{n=1}^{+\infty}\hat{h}_n(x)l_j(h_n) \tag{13}$$

for a.e. $x\in X$.

For each $k\in\mathbb{N}$, let us consider the real matrix $A_k\in\mathbb{R}^{d,k}$ defined by

$$A_k:=(l_i(h_p))_{\substack{i=1,\dots,d\\ p=1,\dots,k}}$$

and the function $\phi_k\in C_b^2(\mathbb{R}^k)$ defined by

$$\phi_k(\xi):=\varphi(A_k\xi),\quad \xi\in\mathbb{R}^k.$$

Let $f_k\in\Sigma$ defined by

$$f_k(x):=\phi_k(\hat{h}_1(x),\dots,\hat{h}_k(x)),\quad \xi\in\mathbb{R}^k.$$

It follows by definition that

$$\begin{aligned}f_k(x)&=\varphi\left(\sum_{n=1}^{k}\hat{h}_n(x)l_1(h_n),\dots,\sum_{n=1}^{k}\hat{h}_n(x)l_d(h_n)\right)=\\&=:\varphi\left(\sum_{n=1}^{k}\hat{h}_n(x)l(h_n)\right),\end{aligned}$$

where we set $l(h_n) := (l_1(h_n), \dots, l_d(h_n)) \in \mathbb{R}^d$. From (13) and the continuity of φ we have $f_k \to f$ a.e. in X and, trivially, $|f_k(x)| \le \|\varphi\|_\infty$. An application of the Lebesgue Dominated Convergence Theorem proves that $f_k \to f$ in $L^p(X, \gamma)$.

Using the chain rule we obtain, for all $j \in \mathbb{N}$

$$[\nabla_H f_k(x), h_j]_H = \partial_j f_k(x) = \begin{cases} \displaystyle\sum_{i=1}^d \frac{\partial\varphi}{\partial\xi_i} \left(\sum_{n=1}^k \hat{h}_n(x) l(h_n) \right) l_i(h_j) & j \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\frac{\partial\varphi}{\partial\xi_i}$ is continuous for every $i = 1, \dots, d$, we have that

$$\partial_j f_k(x) \to \sum_{i=1}^d \frac{\partial\varphi}{\partial\xi_i} (l_1(x), \dots, l_d(x))\, l_i(h_j) = \partial_j f(x)$$

a.e. in X as $k \to +\infty$. Observing that

$$|\partial_j f(x)| \le \|D\varphi\|_\infty \sum_{i=1}^d l_i(h_j)$$

and applying the Dominated Convergence Theorem, we get $\nabla_H f_k \to \nabla_H f$ in $L^p(X, \gamma; H)$.

Let us turn now to the second derivatives. As shown in Lecture 10, pag. 130, by a standard calculus routine we obtain, for $k, s, t \in \mathbb{N}$:

$$[D_H^2 f_k(x) h_s, h_t]_H = \begin{cases} \displaystyle\sum_{i,j=1}^d \frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j} \left(\sum_{n=1}^k \hat{h}_n(x) l(h_n) \right) l_i(h_s) l_j(h_t) & s, t \le k, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$[D_H^2 f(x) h_s, h_t]_H = \sum_{i,j=1}^d \frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j} (l_1(x), \dots, l_d(x))\, l_i(h_s) l_j(h_t).$$

So we have

$$\begin{aligned} &\|D_H^2 f_k(x) - D_H^2 f(x)\|_{\mathcal{H}}^2 = \\ &= \sum_{s,t=1}^{+\infty} \left[\sum_{i,j=1}^d \left(\frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j} \left(\sum_{n=1}^k \hat{h}_n(x) l(h_n) \right) \mathbb{1}_{\{1,\dots,k\}}(s) \mathbb{1}_{\{1,\dots,k\}}(t) + \right.\right. \\ &\left.\left. - \frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j} (l_1(x), \dots, l_d(x)) \right) l_i(h_s) l_j(h_t) \right]^2. \end{aligned} \tag{14}$$

Applying Hölder inequality

$$\|D^2_H f_k(x) - D^2_H f(x)\|^2_{\mathcal{H}} \le \sum_{s,t=1}^{+\infty} \left[\left(2\|D^2\varphi\|_\infty\right)^2 \sum_{i=1}^{d} (l_i(h_s))^2 \sum_{j=1}^{d} (l_j(h_t))^2 \right]$$

which is, recalling $\|R_\gamma l_i\|^2_H = \sum_s (l_i(h_s))^2$,

$$\|D^2_H f_k(x) - D^2_H f(x)\|_{\mathcal{H}} \le 2\|D^2\varphi\|_\infty \sum_{i=1}^{d} \|R_\gamma l_i\|^2_H.$$

Recalling that $\varphi \in C^2(\mathbb{R}^d)$ and using again Lebesgue Theorem to pass to the limit as $k \to +\infty$ inside the sum in (14), we have $\|D^2_H f_k(x) - D^2_H f(x)\|_{\mathcal{H}} \to 0$ a.e. in X. Finally, another application of the Dominated Convergence Theorem proves that $D^2_H f_k \to D^2_H f$ in $L^p(X, \gamma; \mathcal{H})$.

Combining last convergence result with the previous ones, we get $f_k \to f$ in $W^{2,p}(X, \gamma)$, which proves the required assertion.

□

Exercise 13.4.

(i) With the help of Proposition 10.1.2, show that if $f \in W^{1,p}(X, \gamma)$ with $p \in [1, +\infty)$ is such that $\nabla_H f = 0$ a.e., then f is a.e. constant.

(ii) Use point (i) to show that for every $p \in (1, +\infty)$ the kernel of L_p consists of the constant functions.

Solution. Let us fix an orthonormal basis $\{h_k : k \in \mathbb{N}\}$ of H, $h_k = R_\gamma \hat{h}_k$ with $\hat{h}_k \in j(X^*)$ for all $k \in \mathbb{N}$. Given $f \in L^p(X, \gamma)$, let us consider the conditional expectation of f with respect to the σ-algebra Σ_n generated by the random variables $\hat{h}_1, \dots, \hat{h}_n$ (see Proposition 7.4.1), whose expression is

$$\mathbb{E}_n f(x) = \int_X f(P_n x + (I - P_n)y)\, \gamma(dy) =: \overline{\mathbb{E}_n f}(\hat{h}_1(x), \dots, \hat{h}_n(x)),$$

where

$$\overline{\mathbb{E}_n f}(\xi) := \int_X f\left(\sum_{k=1}^{n} \xi_k h_k + (I - P_n)y\right) \gamma(dy), \quad \xi \in \mathbb{R}^n.$$

Let us preliminarily prove that, given a cylindrical function f of the form $f(x) = \overline{f}(\hat{h}_1(x), \dots, \hat{h}_d(x))$, we have that $f \in W^{1,p}(X, \gamma)$ if and only if $\overline{f} \in$

$W^{1,p}(\mathbb{R}^d, \gamma_d)$, with

$$\partial_j f(x) = \begin{cases} \dfrac{\partial \overline{f}}{\partial \xi_j}\Big(\hat{h}_1(x), \dots, \hat{h}_d(x)\Big) & j \leq d, \\ \\ 0 & j > d. \end{cases} \qquad \text{a.e. in } X. \tag{15}$$

Firstly, if $\overline{f} \in W^{1,p}(\mathbb{R}^d, \gamma_d)$, then by definition there exist a sequence $(\overline{f}_n)_{n\in\mathbb{N}} \subset C_b^1(\mathbb{R}^d)$ such that $\overline{f}_n \to \overline{f}$ in $W^{1,p}(\mathbb{R}^d, \gamma_d)$. Recalling $\gamma_d = \gamma \circ (\hat{h}_1, \dots, \hat{h}_d)^{-1}$, we have immediately that $f_n := \overline{f}_n \circ (\hat{h}_1, \dots, \hat{h}_d) \to \overline{f} \circ (\hat{h}_1, \dots, \hat{h}_d) = f$ in $W^{1,p}(X, \gamma)$, namely the first implication required.

Conversely, if $f \in W^{1,p}(X, \gamma)$, we can find a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^1(X)$ such that $f_n \to f$ in $W^{1,p}(X, \gamma)$. By Proposition 10.1.2, we have that

$$\mathbb{E}_d f_n \to \mathbb{E}_d f = f \qquad \text{in } W^{1,p}(X, \gamma). \tag{16}$$

Now, using the representation formula, we have

$$\mathbb{E}_d f_n(x) = \int_X f_n(P_d x + (I - P_d)y)\, \gamma(dy) =: \overline{f}_{n,d}(\hat{h}_1(x), \dots, \hat{h}_d(x)),$$

where

$$\overline{f}_{n,d}(\xi) := \int_X f_n\left(\sum_{k=1}^{d} \xi_k h_k + (I - P_d)y\right) \gamma(dy), \quad \xi \in \mathbb{R}^n. \tag{17}$$

By (16), we have $\overline{f}_{n,d} \circ (\hat{h}_1, \dots, \hat{h}_d) \to \overline{f} \circ (\hat{h}_1, \dots, \hat{h}_d) = f$ in $W^{1,p}(X, \gamma)$, i.e. $\overline{f}_{n,d} \to \overline{f}$ in $W^{1,p}(\mathbb{R}^d, \gamma_d)$. So the required equivalence is proved and formula (15) follows as in Proposition 10.1.2.

Now we are ready to prove statements (i) and (ii).

(i) Let us fix $f \in W^{1,p}(X, \gamma)$, with $p \in [1, +\infty)$, such that $\nabla_H f = 0$ a.e. in X. From Proposition 10.1.2 we have $\mathbb{E}_n f \in W^{1,p}(X, \gamma)$ and

$$\nabla_H(\mathbb{E}_n f) = \mathbb{E}_n(P_n \nabla_H f) \qquad \text{a.e. in } X. \tag{18}$$

Let $\overline{f}_n \in W^{1,p}(\mathbb{R}^n, \gamma_n)$ as in (17) such that

$$\mathbb{E}_n f(x) = \overline{f}_n(\hat{h}_1(x), \dots, \hat{h}_n(x)).$$

Recalling Proposition 9.1.6, we have $\overline{f}_n \in W^{1,p}_{loc}(\mathbb{R}^n)$and, from (18) and (15), $\nabla \overline{f}_n = 0$ a.e. in $\mathbb{R}^n$; so there exists $a_n \in \mathbb{R}$ such that $\overline{f}_n = a_n$ a.e. in $\mathbb{R}^n$. This fact implies that $\mathbb{E}_n f = a_n$ a.e. in X. Recalling that

$$f = \lim_{n\to+\infty} \mathbb{E}_n f = \lim_{n\to+\infty} a_n$$

in $W^{1,p}(X, \gamma)$, we conclude that f is constant a.e. in X.

(ii) Fixed $p > 1$, every a.e. constant function belongs obviously to $D(L_p)$ and for all $c \in \mathbb{R}$ it holds $L_p c = 0$.

Let now $f \in D(L_p)$ such that $L_p f = 0$; we claim that f is constant. To prove this assertion, let us observe that

$$\frac{d}{dt}T(t)f = L_p T(t)f = T(t)L_p f = 0 \tag{19}$$

for every $t \geq 0$. So $t \mapsto T(t)f$ is constant in $[0, +\infty)$, then $T(t)f = T(0)f = f$ for every $t \geq 0$. Then, for every $t > 0$, $f = T(t)f \in W^{1,p}(X, \gamma)$ by Proposition 12.1.6(ii). Moreover, by Proposition 12.1.6(i), we have for every $t > 0$

$$\nabla_H f = \nabla_H T(t)f = e^{-t}T(t)(\nabla_H f),$$

so

$$|\nabla_H f|_H \leq e^{-t}|\nabla_H f|_H.$$

Since the second addendum tends to zero as t goes to $+\infty$, it results $|\nabla_H f|_H = 0$. Then $\nabla_H f = 0$ a.e. in X, so, by part (i), f is a.e. constant in X and assertion (ii) is proved.

□