Lecture 12

The Ornstein–Uhlenbeck semigroup

All of us know the importance of the Laplacian operator Δ and of the heat semigroup,

$$\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(x), \qquad T(t)f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y)dy, \quad t > 0,$$

that serve as prototypes for elliptic differential operators and semigroups of operators, respectively. Choosing $X = L^2(\mathbb{R}^d, \lambda_d)$, the Laplacian $\Delta : D(\Delta) = W^{2,2}(\mathbb{R}^d, \lambda_d) \to X$ is the infinitesimal generator of $T(t)$, namely given any $f \in X$, there exists the limit $\lim_{t\to 0^+}(T(t)f - f)/t$ if and only if $f \in W^{2,2}(\mathbb{R}^d, \lambda_d)$, and in this case the limit is Δf. The $W^{2,2}$ norm is equivalent to the graph norm. Moreover, the realization of the Laplacian in X is the operator associated with the quadratic form

$$\mathcal{Q}(u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \; dx, \quad u,\; v \in W^{1,2}(\mathbb{R}^d, \lambda_d).$$

This means that $D(\Delta)$ consists precisely of the elements $u \in W^{1,2}(\mathbb{R}^d, \lambda_d)$ such that the function $W^{1,2}(\mathbb{R}^d, \lambda_d) \to \mathbb{R}$, $\varphi \mapsto \int_{\mathbb{R}^d} \nabla u \cdot \nabla \varphi \, dx$ has a linear bounded extension to the whole X, namely there exists $g \in L^2(\mathbb{R}^d, \lambda_d)$ such that

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla \varphi \; dx = \int_{\mathbb{R}^d} g\, \varphi \, dx, \quad \varphi \in W^{1,2}(\mathbb{R}^d, \lambda_d),$$

and in this case $g = -\Delta u$. If $u \in W^{2,2}(\mathbb{R}^d, \lambda_d)$, the above formula follows just integrating by parts, and it is the basic formula that relates the Laplacian and the gradient. Moreover, for $u \in W^{2,2}(\mathbb{R}^d, \lambda_d)$ we have

$$\Delta u = \operatorname{div} \nabla u,$$

where the divergence div is (minus) the adjoint of the gradient $\nabla : W^{1,2}(\mathbb{R}^d, \lambda_d) \to L^2(\mathbb{R}^d, \lambda_d; \mathbb{R}^d)$, and for a vector field $v \in W^{1,2}(\mathbb{R}^d, \lambda_d; \mathbb{R}^d)$ it is given by $\sum_{i=1}^{d} \partial v_i / \partial x_i$.

In this lecture and in the next ones we introduce the Ornstein–Uhlenbeck operator and the Ornstein–Uhlenbeck semigroup, that play the role of the Laplacian and of the heat semigroup if the Lebesgue measure is replaced by the standard Gaussian measure γ_d, and that have natural extensions to our infinite dimensional setting (X, γ, H). As before, X is a separable Banach space endowed with a centred nondegenerate Gaussian measure γ, and H is the relevant Cameron–Martin space.

12.1 The Ornstein–Uhlenbeck semigroup

In this section we define the Ornstein–Uhlenbeck semigroup; we start by defining it in the space of the bounded continuous functions and then we extend it to $L^p(X, \gamma)$, for every $p \in [1, +\infty)$.

The Ornstein–Uhlenbeck semigroup in $C_b(X)$ is defined as follows: $T(0) = I$, and for $t > 0$, $T(t)f$ is defined by the Mehler formula

$$T(t)f(x) := \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\gamma(dy). \tag{12.1.1}$$

We list some properties of the family of operators $\{T(t) : t \geq 0\}$.

Proposition 12.1.1. *$\{T(t) : t \geq 0\}$ is a contraction semigroup in $C_b(X)$. Moreover, for every $f \in C_b(X)$ we have*

$$\int_X T(t)f \, d\gamma = \int_X f \, d\gamma, \quad t > 0. \tag{12.1.2}$$

Proof. First of all we notice that $|T(t)f(x)| \leq \|f\|_\infty$ for every $x \in X$ and $t \geq 0$. The fact that $T(t)f \in C_b(X)$ follows by Dominated Convergence Theorem. So, $T(t) \in \mathcal{L}(C_b(X))$ and $\|T(t)\|_{\mathcal{L}(C_b(X))} \leq 1$. Taking $f \equiv 1$, we have $T(t)f \equiv 1$ so that $\|T(t)\|_{\mathcal{L}(C_b(X))} = 1$ for every $t \geq 0$.

Let us prove that $\{T(t) : t \geq 0\}$ is a semigroup. For every $f \in C_b(X)$ and $t, s > 0$ we have

$$\begin{aligned}
(T(t)(T(s)f))(x) &= \int_X (T(s)f)(e^{-t}x + \sqrt{1 - e^{-2t}}y)\gamma(dy) \\
&= \int_X \int_X f(e^{-s}(e^{-t}x + \sqrt{1 - e^{-2t}}y) + \sqrt{1 - e^{-2s}}z)\gamma(dy)\gamma(dz).
\end{aligned}$$

Setting now $\Phi(y, z) = e^{-s} \frac{\sqrt{1-e^{-2t}}}{\sqrt{1-e^{-2t-2s}}} y + \frac{\sqrt{1-e^{-2s}}}{\sqrt{1-e^{-2t-2s}}} z$, and using Proposition 2.2.7(iv), we get

$$\begin{aligned}
(T(t)(T(s)f))(x) &= \int_X \int_X f(e^{-s-t}x + \sqrt{1 - e^{-2t-2s}}\Phi(y, z))\gamma(dy)\gamma(dz) \\
&= \int_X f(e^{-t-s}x + \sqrt{1 - e^{-2t-2s}}\xi)((\gamma \otimes \gamma) \circ \Phi^{-1})(d\xi) \\
&= \int_X f(e^{-t-s}x + \sqrt{1 - e^{-2t-2s}}\xi)\gamma(d\xi) \\
&= T(t + s)f(x).
\end{aligned}$$

Let us prove that (12.1.2) holds. For any $f \in C_b(X)$ we have

$$\int_X T(t)f\,\gamma(dx) = \int_X \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(dy)\gamma(dx).$$

Setting $\phi(x,y) = e^{-t}x + \sqrt{1-e^{-2t}}y$, we apply Proposition 2.2.7(iv) with any $\theta \in \mathbb{R}$ such that $e^{-t} = \cos\theta$, $\sqrt{1-e^{-2t}} = \sin\theta$, and we get

$$\int_X T(t)f\,d\gamma = \int_X f(\xi)(\gamma\otimes\gamma)\circ\phi^{-1}(d\xi) = \int_X f(\xi)\gamma(d\xi).$$

□

We point out that the semigroup $\{T(t):\ t \geq 0\}$ is not strongly continuous in $C_b(X)$, and not even in the subspace $BUC(X)$ of the bounded and uniformly continuous functions. In fact, we have the following characterisation.

Lemma 12.1.2. *Let $f \in BUC(X)$. Then*

$$\lim_{t\to 0^+} \|T(t)f - f\|_\infty = 0 \Longleftrightarrow \lim_{t\to 0^+} \|f(e^{-t}\cdot) - f\|_\infty = 0.$$

Proof. For every $t > 0$ and $x \in X$ we have

$$(T(t)f - f(e^{-t}\cdot))(x) = \int_X (f(e^{-t}x + \sqrt{1-e^{-2t}}y) - f(e^{-t}x))\gamma(dy)$$

and the right hand side goes to 0, uniformly in X, as $t \to 0^+$. Indeed, given $\varepsilon > 0$ fix $R > 0$ such that $\gamma(X \setminus B(0,R)) \leq \varepsilon$, and fix $\delta > 0$ such that $|f(u) - f(v)| \leq \varepsilon$ for $\|u - v\| \leq \delta$. Then, for every t such that $\sqrt{1-e^{-2t}}R \leq \delta$ and for every $x \in X$ we have

$$\begin{aligned}
&\left|\int_X (f(e^{-t}x + \sqrt{1-e^{-2t}}y) - f(e^{-t}x))\gamma(dy)\right| \\
&= \left|\left(\int_{B(0,R)} + \int_{X\setminus B(0,R)}\right)(f(e^{-t}x + \sqrt{1-e^{-2t}}y) - f(e^{-t}x))\gamma(dy)\right| \\
&\leq \varepsilon + 2\|f\|_\infty\varepsilon.
\end{aligned}$$

□

The simplest counterexample to strong continuity in $BUC(X)$ is one dimensional: see Exercise 12.1.

However, for every $f \in C_b(X)$ the function $(t,x) \mapsto T(t)f(x)$ is continuous in $[0,+\infty)\times X$ by the Dominated Convergence Theorem. In particular,

$$\lim_{t\to 0^+} T(t)f(x) = f(x), \quad \forall x \in X,$$

which is enough for many purposes.

The semigroup $\{T(t):\ t \geq 0\}$ enjoys important smoothing and summability improving properties. The first smoothing property is in the next proposition.

Proposition 12.1.3. *For every $f \in C_b(X)$ and $t > 0$, $T(t)f$ is H-differentiable at every $x \in X$, and we have*

$$[\nabla_H T(t)f(x), h]_H = \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\hat{h}(y)\gamma(dy). \tag{12.1.3}$$

Therefore,

$$|\nabla_H T(t)f(x)|_H \leq \frac{e^{-t}}{\sqrt{1-e^{-2t}}}|h|_H\|f\|_\infty, \quad x \in X. \tag{12.1.4}$$

Proof. Set

$$c(t) := \frac{e^{-t}}{\sqrt{1-e^{-2t}}}.$$

For every $h \in H$ we have

$$\begin{aligned} T(t)f(x+h) &= \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}(c(t)h + y))\gamma(dy) \\ &= \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}z)\exp\Big\{c(t)\hat{h}(z) - c(t)^2\frac{|h|_H^2}{2}\Big\}\,\gamma(dz), \end{aligned}$$

by the Cameron–Martin formula. Therefore, denoting by $l_{t,x}(h)$ the right hand side of (12.1.3),

$$\begin{aligned} &\frac{|T(t)f(x+h) - T(t)f(x) - l_{t,x}(h)|}{|h|_H} \leq \\ &\leq \frac{1}{|h|_H}\int_X \Big| f(e^{-t}x + \sqrt{1-e^{-2t}}y)\Big(\exp\Big\{c(t)\hat{h}(y) - c(t)^2\frac{|h|_H^2}{2}\Big\} - 1\Big) - c(t)\hat{h}(y)\Big|\gamma(dy) \\ &\leq \frac{\|f\|_\infty}{|h|_H}\int_{\mathbb{R}} \Big|\exp\Big\{c(t)\xi - c(t)^2\frac{|h|_H^2}{2}\Big\} - 1 - c(t)\xi\Big|\mathscr{N}(0,|h|_H^2)(d\xi) \\ &= \|f\|_\infty \int_{\mathbb{R}} \Big|\exp\Big\{c(t)|h|_H\eta - c(t)^2\frac{|h|_H^2}{2}\Big\} - 1 - c(t)|h|_H\eta\Big|\mathscr{N}(0,1)(d\eta) \end{aligned}$$

where the right hand side vanishes as $|h|_H \to 0$, by the Dominated Convergence Theorem. This proves (12.1.3). In its turn, (12.1.3) yields

$$|[\nabla_H T(t)f(x), h]_H| \leq c(t)\|f\|_\infty\|\hat{h}\|_{L^1(X,\gamma)} \leq c(t)\|f\|_\infty\|\hat{h}\|_{L^2(X,\gamma)} = c(t)\|f\|_\infty|h|_H$$

for every $h \in H$, and (12.1.4) follows. □

Notice that in the case $X = \mathbb{R}^d$, $\gamma = \gamma_d$, we have $\nabla_H = \nabla$ and formula (12.1.3) reads as

$$D_i T(t)f(x) = \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1-e^{-2t}}y)y_i\,\gamma_d(dy), \quad i = 1,\ldots,d. \tag{12.1.5}$$

Let us consider now more regular functions f.

Proposition 12.1.4. *For every $f \in C_b^1(X)$, $T(t)f \in C_b^1(X)$ for any $t \geq 0$, and its derivative at x is*

$$(T(t)f)'(x)(v) = e^{-t}\int_X f'(e^{-t}x + \sqrt{1-e^{-2t}}y)(v)\,\gamma(dy). \tag{12.1.6}$$

In particular,

$$\frac{\partial}{\partial v}T(t)f(x) = e^{-t}\Big(T(t)\Big(\frac{\partial}{\partial v}f\Big)\Big)(x), \quad v\in X,\ x\in X. \tag{12.1.7}$$

For every $f \in C_b^2(X)$, $T(t)f \in C_b^2(X)$ for any $t \geq 0$, and its second order derivative at x is

$$(T(t)f)''(x)(u)(v) = e^{-2t}\int_X f''(e^{-t}x + \sqrt{1-e^{-2t}}y)(u)(v)\,\gamma(dy), \tag{12.1.8}$$

so that

$$\frac{\partial^2 T(t)f}{\partial u\partial v}(x) = e^{-2t}T(t)\bigg(\frac{\partial^2 f}{\partial u\partial v}\bigg)(x), \quad u,\ v\in X,\ x\in X. \tag{12.1.9}$$

Proof. Fix $t > 0$ and set $\Phi(x,y) = e^{-t}x + \sqrt{1-e^{-2t}}y$. For every x, $v \in X$ we have

$$\begin{aligned}&\bigg|(T(t)f)(x+v) - (T(t)f)(x) - e^{-t}\int_X f'(e^{-t}x + \sqrt{1-e^{-2t}}y)(v)\,\gamma(dy)\bigg|\frac{1}{\|v\|}\\ &\leq \int_X |f(\Phi(x,y)+e^{-t}v) - f(\Phi(x,y)) - f'(\Phi(x,y))(e^{-t}v)|\,\gamma(dy)\frac{1}{\|v\|}.\end{aligned}$$

On the other hand, for every $y \in X$ we have

$$\lim_{v\to 0}\frac{f(\Phi(x,y)e^{-t}v) - f(\Phi(x,y)) - f'(\Phi(x,y))(e^{-t}v)}{\|v\|} = 0,$$

and (see Exercise 12.3)

$$\frac{|f(\Phi(x,y)+e^{-t}v) - f(\Phi(x,y)) - f'(\Phi(x,y))(e^{-t}v)|}{\|v\|} \leq 2e^{-t}\sup_{z\in X}\|f'(z)\|_{X^*},$$

and (12.1.6) follows by the Dominated Convergence Theorem. Formula (12.1.7) is an immediate consequence. The derivative $(T(t)f)'$ is continuous still by the Dominated Convergence Theorem. The verification of (12.1.8) and (12.1.9) for $f \in C_b^2(X)$ follow iterating this procedure (see Exercise 12.4). □

Let us compare Proposition 12.1.3 and Proposition 12.1.4. Proposition 12.1.3 describes a smoothing property of $T(t)$, while Proposition 12.1.4 says that $T(t)$ preserves the spaces $C_b^1(X)$ and $C_b^2(X)$. In general, $T(t)$ regularises only along H and it does not map $C_b(X)$ into $C^1(X)$. If $X = \mathbb{R}^d$ and $\gamma = \gamma_d$ we have $H = \mathbb{R}^d$, this difficulty does not arise, Proposition 12.1.3 says that $T(t)$ maps $C_b(\mathbb{R}^d)$ into $C_b^1(\mathbb{R}^d)$ and in fact one can check that $T(t)$ maps $C_b(\mathbb{R}^d)$ into $C_b^k(\mathbb{R}^d)$ for every $k \in \mathbb{N}$, as we shall see in the next lecture.

If $f \in C_b^1(X)$ we can write $\nabla_H T(t)f(x)$ in two different ways, using (12.1.3) and (12.1.6): for every $h \in H$ we have

$$[\nabla_H T(t)f(x), h]_H = \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\hat{h}(y)\gamma(dy)$$
$$= e^{-t} \int_X f'(e^{-t}x + \sqrt{1-e^{-2t}}y)(h)\,\gamma(dy).$$

We now extend $T(t)$ to $L^p(X,\gamma)$, $1 \le p < \infty$.

Proposition 12.1.5. *Let $t \ge 0$. For every $f \in C_b(X)$ and $p \in [1, +\infty)$ we have*

$$\|T(t)f\|_{L^p(X,\gamma)} \le \|f\|_{L^p(X,\gamma)}. \tag{12.1.10}$$

Hence $\{T(t):\ t \ge 0\}$ is uniquely extendable to a contraction semigroup $\{T_p(t):\ t \ge 0\}$ in $L^p(X,\gamma)$. Moreover

(i) $\{T_p(t):\ t \ge 0\}$ is strongly continuous in $L^p(X,\gamma)$, for every $p \in [1, +\infty)$;

(ii) $T_2(t)$ is self–adjoint and nonnegative in $L^2(X,\gamma)$ for every $t > 0$;

(iii) γ is an invariant measure for $\{T_p(t):\ t \ge 0\}$.

Proof. For every $f \in C_b(X)$, $t > 0$ and $x \in X$ the Hölder inequality yields

$$|T(t)f(x)|^p \le \int_X |f(e^{-t}x + \sqrt{1-e^{-2t}}y)|^p d\gamma = T(t)(|f|^p)(x).$$

Integrating over X and using (12.1.2) we obtain

$$\int_X |T(t)f|^p d\gamma \le \int_X T(t)(|f|^p)\, d\gamma = \int_X |f|^p d\gamma.$$

Since $C_b(X)$ is dense in $L^p(X,\gamma)$, $T(t)$ has a unique bounded extension $T_p(t)$ to the whole $L^p(X,\gamma)$, such that $\|T_p(t)\|_{\mathcal{L}(L^p(X,\gamma))} \le 1$. In fact, taking $f \equiv 1$, $T_p(t)f \equiv 1$ so that $\|T_p(t)\|_{\mathcal{L}(L^p(X,\gamma))} = 1$.

Let us prove that $\{T_p(t):\ t \ge 0\}$ is strongly continuous. We already know that for $f \in C_b(X)$ we have $\lim_{t\to 0^+} T(t)f(x) = f(x)$ for every $x \in X$, and moreover $|T(t)f(x)| \le \|f\|_\infty$ for every x. By the Dominated Convergence Theorem, $\lim_{t\to 0^+} T(t)f = f$ in $L^p(X,\gamma)$. Since $C_b(X)$ is dense in $L^p(X,\gamma)$ and $\|T_p(t)\|_{\mathcal{L}(L^p(X,\gamma))} = 1$ for every t, then $\lim_{t\to 0^+} T_p(t)f = f$ for every $f \in L^p(X,\gamma)$.

Let us prove statement (ii). Let f, $g \in C_b(X)$, $t > 0$. Then

$$\langle T(t)f, g\rangle_{L^2(X,\gamma)} = \int_X \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)g(x)\gamma(dy)\gamma(dx)$$

and setting $u = e^{-t}x + \sqrt{1-e^{-2t}}y$, $v = -\sqrt{1-e^{-2t}}x + e^{-t}y$, $(u,v) =: R(x,y)$, using Proposition 2.2.7(iii) and the fact that γ is centred we get

$$\begin{aligned}\langle T(t)f, g\rangle_{L^2(X,\gamma)} &= \int_{X\times X} f(u)g(e^{-t}u - \sqrt{1-e^{-2t}}v)((\gamma\otimes\gamma)\circ R^{-1})(d(u,v))\\ &= \int_X\int_X f(u)g(e^{-t}u - \sqrt{1-e^{-2t}}v)\gamma(dv)\gamma(du)\\ &= \int_X\int_X f(u)g(e^{-t}u + \sqrt{1-e^{-2t}}v)\gamma(dv)\gamma(du)\\ &= \langle f, T(t)g\rangle_{L^2(X,\gamma)}.\end{aligned}$$

Approximating any f, $g \in L^2(X,\gamma)$ by elements of $C_b(X)$, we obtain $\langle T_2(t)f, g\rangle_{L^2(X,\gamma)} = \langle f, T_2(t)g\rangle_{L^2(X,\gamma)}$.

Still for every $f \in L^2(X,\gamma)$ and $t > 0$ we have

$$\langle T_2(t)f, f\rangle_{L^2(X,\gamma)} = \langle T_2(t/2)T_2(t/2)f, f\rangle_{L^2(X,\gamma)} = \langle T_2(t/2)f, T_2(t/2)f\rangle_{L^2(X,\gamma)} \ge 0.$$

Statement (iii) is an immediate consequence of (12.1.2). □

To simplify some statements we extend the Ornstein–Uhlenbeck semigroup $T(t)$ to vector valued functions. For $v \in C_b(X;H)$ and $t > 0$ we set

$$T(t)v(x) = \int_X v(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(dy), \quad t > 0,\ x \in X.$$

By Remark 9.2.6, for every orthonormal basis $\{h_i:\ i \in \mathbb{N}\}$ of H we have

$$T(t)v(x) = \sum_{i=1}^{\infty} T(t)([v(\cdot), h_i]_H)(x)h_i.$$

Using Proposition 9.2.5(i) we get the estimate

$$|T(t)v(x)|_H \le \int_X |v(e^{-t}x + \sqrt{1-e^{-2t}}y)|_H\,\gamma(dy) = (T(t)(|v|_H))(x), \quad x \in X. \tag{12.1.11}$$

Notice that the right hand side concerns the scalar valued function $|v|_H$. Raising to the power p, integrating over X and recalling (12.1.2), we obtain

$$\|T(t)v\|_{L^p(X,\gamma;H)} \le \|v\|_{L^p(X,\gamma;H)}, \quad t \ge 0. \tag{12.1.12}$$

As in the case of real valued functions, since $C_b(X;H)$ is dense in $L^p(X,\gamma;H)$, estimate (12.1.12) allows to extend $T(t)$ to a bounded (contraction) operator in $L^p(X,\gamma;H)$, called $T_p(t)$. We will not develop the theory for vector valued functions, but we shall use this notion to write some formulae in a concise way, see e.g. (12.1.13).

L^p gradient estimates for $T_p(t)f$ are provided by the next proposition.

Proposition 12.1.6. *Let* $1 \le p < \infty$.

(i) For every $f \in W^{1,p}(X,\gamma)$ and $t>0$, $T_p(t)f \in W^{1,p}(X,\gamma)$ and

$$\nabla_H T_p(t)f = e^{-t}T_p(t)(\nabla_H f), \tag{12.1.13}$$

$$\|T_p(t)f\|_{W^{1,p}(X,\gamma)} \le \|f\|_{W^{1,p}(X,\gamma)}. \tag{12.1.14}$$

(ii) If $p>1$, for every $f \in L^p(X,\gamma)$ and $t>0$, $T_p(t)f \in W^{1,p}(X,\gamma)$ and

$$\int_X |\nabla_H T_p(t)f(x)|_H^p d\gamma \le c(t,p)^p \int_X |f|^p d\gamma, \tag{12.1.15}$$

with $c(t,p) = \bigg(\int_{\mathbb{R}} |\xi|^{p'} \mathscr{N}(0,1)(d\xi)\bigg)^{1/p'} e^{-t}/\sqrt{1-e^{-2t}}$.

Proof. (i). Let $f \in C^1_b(X)$. By Proposition 12.1.4, $T(t)f \in C^1_b(X)$ and $(\partial_h T(t)f)(x) = e^{-t}(T(t)(\partial_h f))(x)$ for every $h \in H$, namely $[(\nabla_H T(t)f)(x), h]_H = e^{-t}T(t)([\nabla_H f, h]_H)(x)$. Therefore, $|(\nabla_H T(t)f)(x)|_H \le e^{-t}T(t)(|\nabla_H f|_H)(x)$, for every $x \in X$. Consequently,

$$|\nabla_H T(t)f(x)|_H^p \le e^{-tp}(T(t)(|\nabla_H f|_H)(x))^p \le e^{-tp}(T(t)(|\nabla_H f|_H^p)(x))$$

and integrating we obtain

$$\int_X |\nabla_H T(t)f(x)|_H^p d\gamma \le e^{-tp}\int_X T(t)(|\nabla_H f|_H)^p d\gamma = e^{-tp}\int_X |\nabla_H f|_H^p d\gamma,$$

so that

$$\begin{aligned}\|T(t)f\|_{W^{1,p}(X,\gamma)} &= \|T(t)f\|_{L^p(X,\gamma)} + \|\, |\nabla_H T(t)f|_H \,\|_{L^p(X,\gamma)} \\ &\le \|f\|_{L^p(X,\gamma)} + \|\, |\nabla_H f|_H \,\|_{L^p(X,\gamma)} = \|f\|_{W^{1,p}(X,\gamma)}.\end{aligned}$$

Since $C^1_b(X)$ is dense in $W^{1,p}(X,\gamma)$, (12.1.14) follows.

(ii) Let $f \in C_b(X)$. By Proposition 12.1.3, $T(t)f$ is H-differentiable at every x, and we have

$$|\nabla_H T(t)f(x)|_H = \sup_{h\in H,\, |h|_H=1} |[\nabla_H T(t)f(x), h]_H|.$$

Let us estimate $|[\nabla_H T(t)f(x), h]_H| = |l_{t,x}(h)|$ (where $l_{t,x}(h)$ is as in the proof of Proposition 12.1.3), for $|h|_H = 1$, using formula (12.1.3). We have

$$\begin{aligned}|l_{t,x}(h)| &\le \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\int_X |f(e^{-t}x + \sqrt{1-e^{-2t}}y)|\hat{h}(y)\gamma(dy) \\ &\le \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\bigg(\int_X |f(e^{-t}x+\sqrt{1-e^{-2t}}y)|^p\gamma(dy)\bigg)^{1/p}\bigg(\int_X |\hat{h}|^{p'}d\gamma\bigg)^{1/p'} \\ &= \frac{e^{-t}}{\sqrt{1-e^{-2t}}}(T(t)(|f|^p)(x))^{1/p}\bigg(\int_{\mathbb{R}} |\xi|^{p'}\mathscr{N}(0,1)(d\xi)\bigg)^{1/p'}.\end{aligned}$$

By the invariance property (12.1.2) of γ,

$$\int_X |\nabla_H T(t)f(x)|_H^p\,\gamma(dx) \le \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\right)^p \int_X (T(t)(|f|^p)\,d\gamma\left(\int_{\mathbb{R}} |\xi|^{p'}\mathscr{N}(0,1)(d\xi)\right)^{p/p'}$$
$$= \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\right)^p \int_X |f|^p d\gamma\left(\int_{\mathbb{R}} |\xi|^{p'}\mathscr{N}(0,1)(d\xi)\right)^{p/p'}.$$

Therefore, $T(t)f \in W^{1,p}(X,\gamma)$ and estimate (12.1.15) holds for every $f \in C_b(X)$. Since $C_b(X)$ is dense in $L^p(X,\gamma)$, the statement follows. □

Note that the proof of (ii) fails for $p = 1$, since for every $h \in H$ the function $\hat{h}$ does not belong to L^∞, and the constant $c(t,p)$ in estimate (12.1.15) blows up as $p \to 1$. Indeed, $T(t)$ does not map $L^1(X,\gamma)$ into $W^{1,1}(X,\gamma)$ continuously for $t > 0$, even in the simplest case $X = \mathbb{R}$, $\gamma = \gamma_1$ (see for instance [MPP, Corollary 5.1]).

12.2 Exercises

Exercise 12.1. Let $X = \mathbb{R}$ and set $f(x) = \sin x$. Prove that $T(t)f$ does not converge uniformly to f in $\mathbb{R}$ as $t \to 0$.

Exercise 12.2. Show that the argument used in Proposition 12.1.5 to prove that $T(t)$ is self–adjoint in $L^2(X,\gamma)$ implies that $T_{p'}(t) = T_p(t)^*$ for $p \in (1,+\infty)$ with $1/p + 1/p' = 1$.

Exercise 12.3. Prove that for every $f \in C^1(X)$ and for every x, $y \in X$ we have

$$f(y) - f(x) = \int_0^1 f'(\sigma y + (1-\sigma)x)(y-x)\,d\sigma,$$

so that, if $f \in C^1_b(X)$,

$$|f(y) - f(x)| \le \sup_{z\in X} \|f'(z)\|_{X'}\|y-x\|.$$

Exercise 12.4. Prove that for every $f \in C^2_b(X)$ and $t > 0$, $T(t)f \in C^2_b(X)$ and (12.1.8), (12.1.9) hold.

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