ISem 2015/16: Solutions to the Exercises of Lecture 11

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Exercise 1. *Let $\mathbb{R}$ be endowed with the Lebesgue measure λ_1, and let $f\colon [a,b] \to X$ be a continuous function. Prove that it is Bochner integrable, that*

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{n\to\infty} \sum_{i=1}^n f(\tau_i)\frac{b-a}{n}$$

for any choice of $\tau_1 \in [a+(b-a)(i-1)/n, a+(b-a)i/n]$, $i = 1,\dots,n$ (the sums in this approximation are the usual Riemann sums in the real-valued case) and that, setting

$$F(t) = \int_a^t f(s)\,\mathrm{d}s\,, \quad a \le t \le b\,,$$

the function F is continuously differentiable, with

$$F'(t) = f(t)\,, \quad a \le t \le b\,.$$

Proof. By Proposition 9.2.4 we know that f is Bochner integrable if and only if

$$\int_a^b \|f(t)\|_X \,\mathrm{d}\lambda_1(t) < \infty\,.$$

Since f is continuous this condition holds. Furthermore every absolute Riemann integrable function is Lebesgue integrable and the two integrals coincide. Hence the proof of the second claim is literally the same as the one of the Riemann integral.

For the last claim, let $t_0 \in [a,b)$ and $0 < h < b - t_0$, then

$$\frac{F(t_0+h)-F(t_0)}{h} = \frac{1}{h}\int_{t_0}^{t_0+h} f(t)\,\mathrm{d}t\,.$$

The continuity of f implies

$$\left\|\frac{F(t_0+h)-F(t_0)}{h} - f(t_0)\right\| \le \frac{1}{h}\int_{t_0}^{t_0+h} \|f(t)-f(t_0)\|\,\mathrm{d}t \le \max_{s\in[t_0,t_0+h]} \|f(s)-f(t_0)\|\,,$$

where the last term tends to 0 as $h \to 0^+$. Hence $F'_+(t_0) = f(t_0)$. In the same way one obtains

$$\forall\, t_0 \in (a,b]\colon\, F'_-(t_0) = f(t_0)\,.$$

Thus F is continuously differentiable on $[a,b]$ and $F' = f$. ■

Exercise 2. *Prove that if $u \in C([0,+\infty); D(L)) \cap C^1([0,+\infty); X)$ is a solution of problem (11.1.3), then for $t > 0$ the function $v(s) = T(t-s)u(s)$ is continuously differentiable in $[0,t]$ and it verifies $v'(s) = -T(t-s)Lu(s) + T(t-s)u'(s) = 0$ for $0 \le s \le t$.*

Proof. Since L is the generator of $\{T_t\}$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)f = T(t)Lf \text{ for } f \in D(L)$$

by Proposition 11.1.5. Hence

$$\begin{aligned} v(s) &= T(t-s)u(s)\,, \\ v'(s) &= -T(t-s)Lu(s) + T(t-s)u'(s)\,, \end{aligned}$$

by application of the product and chain rule. As $u'(s) = Lu(s)$, this yields the claim. ■

Exercise 3. *Let $L\colon D(L) \subset X \to X$ be a linear operator. Prove that if $\rho(L) \neq \varnothing$ then L is closed.*

Proof. By definition

$$\rho(L) = \{\lambda \in \mathbb{C} \mid \lambda I - L \text{ invertible and } (\lambda I - L)^{-1} \in \mathcal{L}(X)\}\,.$$

Since $\rho(L) \neq \varnothing$, we can define $\widetilde{L}^{-1} \coloneqq (\lambda I - L)^{-1} \in \mathcal{L}(X)$ for some $\lambda \in \rho(L)$. Let $\{x_n\}$ be a sequence in X with $x_n \to x$ and $\widetilde{L}x_n \to y$ for some $y \in X$. We want to show that $x \in D(\widetilde{L}) = D(L)$ and $y = \widetilde{L}x$. As $\widetilde{L}$ has a bounded inverse, we set $x \coloneqq \widetilde{L}^{-1}y$ and get $\widetilde{L}x = y$. This shows that $\widetilde{L}$ is closed. As I is closed and $L = \lambda I - \widetilde{L}$, the operator L is closed as well. ■

Exercise 4. *Prove the resolvent identity*

$$R(\lambda, L) - R(\mu, L) = (\mu - \lambda) R(\lambda, L) R(\mu, L)$$

for all λ, $\mu \in \rho(L)$.

Proof. Straightforward calculations yield

$$\begin{aligned} R(\lambda, L) - R(\mu, L) &= (\lambda I - L)^{-1} - (\mu I - L)^{-1} \\ &= (\lambda I - L)^{-1}\Big((\mu I - L) - (\lambda I - L)\Big)(\mu I - L)^{-1} \\ &= (\mu - \lambda)(\lambda I - L)^{-1}(\mu I - L)^{-1} \\ &= (\mu - \lambda) R(\lambda, L) R(\mu, L)\,. \end{aligned}$$

Thus the resolvent identity holds for all $\lambda, \mu \in \rho(L)$. ■

Exercise 5. *Prove that in Hilbert spaces the dissipativity condition*

$$\forall \lambda > 0 \, \forall x \in D(L) \colon \|(\lambda I - L)x\| \geq \lambda \|x\| \tag{1}$$

is equivalent to

$$\forall x \in D(L) \colon \operatorname{Re}\langle Lx, x\rangle \leq 0$$

for a linear operator $(L, D(L))$.

Proof. Assume the dissipativity condition (1) holds. For $x \in D(L)$, write

$$\begin{aligned}\|(\lambda I - L)x\|^2 &= \langle \lambda x - Lx, \lambda x - Lx\rangle = \lambda^2\|x\|^2 - \lambda\langle x, Lx\rangle - \lambda\langle Lx, x\rangle + \|Lx\|^2 \\ &= \lambda^2\|x\|^2 - 2\lambda\operatorname{Re}\langle Lx, x\rangle + \|Lx\|^2 \geq \lambda^2\|x\|^2 \,,\end{aligned}$$

where the last inequality follows from (1). Hence $\|Lx\|^2 \geq 2\lambda\operatorname{Re}\langle Lx, x\rangle$ and

$$2\operatorname{Re}\langle Lx, x\rangle \leq \lim_{\lambda\to\infty} \frac{1}{\lambda}\|Lx\|^2 = 0 \,,$$

which implies $\operatorname{Re}\langle Lx, x\rangle \leq 0$.

Now assume $\operatorname{Re}\langle Lx, x\rangle \leq 0$. We have already seen that

$$\|(\lambda I - L)x\|^2 = \lambda^2\|x\|^2 - 2\lambda\operatorname{Re}\langle Lx, x\rangle + \|Lx\|^2$$

holds for all $x \in D(L)$. The last two terms are nonnegative, thus

$$\|(\lambda I - L)x\|^2 \geq \lambda^2\|x\|^2 \,,$$

which concludes the proof. ∎

Exercise 6. *Let* $\{T(t) \mid t \geq 0\}$ *be a bounded strongly continuous semigroup. Prove that the norm*

$$|x| := \sup_{t\geq 0}\|T(t)x\|$$

is equivalent to $\|\cdot\|$ *and that* $T(t)$ *is contractive on* $(X, |\cdot|)$.

Proof. Since the semigroup is bounded by M, $|x| \leq M\|x\|$ holds. On the other hand

$$\|x\| = \|T(0)x\| \leq \sup_{t\geq 0}\|T(t)x\| = |x| \,.$$

This shows that the two norms are equivalent. By the semigroup property, we get

$$|T(t)x| = \sup_{s\geq 0}\|T(s)T(t)x\| = \sup_{s\geq 0}\|T(s+t)x\| \leq \sup_{u\geq 0}\|T(u)x\| = |x| \,.$$

Thus $\{T(t)\}$ is a contractive semigroup on $(X, |\cdot|)$. ∎