Lecture 11

Semigroups of Operators

In this Lecture we gather a few notions on one-parameter semigroups of linear operators, confining to the essential tools that are needed in the sequel. As usual, X is a real or complex Banach space, with norm $\|\cdot\|$. In this lecture Gaussian measures play no role.

11.1 Strongly continuous semigroups

Definition 11.1.1. *Let $\{T(t) : t \geq 0\}$ be a family of operators in $\mathcal{L}(X)$. We say that it is a* semigroup *if*

$$T(0) = I, \quad T(t+s) = T(t)T(s) \;\; \forall\, t, s \geq 0.$$

A semigroup is called strongly continuous *(or C_0-semigroup) if for every $x \in X$ the function $T(\cdot)x : [0, \infty) \to X$ is continuous.*

Let us present the most elementary properties of strongly continuous semigroups.

Lemma 11.1.2. *Let $\{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a semigroup. The following properties hold:*

(a) if there exist $\delta > 0$, $M \geq 1$ such that

$$\|T(t)\| \leq M, \;\; 0 \leq t \leq \delta,$$

then, setting $\omega = (\log M)/\delta$ we have

$$\|T(t)\| \leq M e^{\omega t}, \;\; t \geq 0. \tag{11.1.1}$$

Moreover, for every $x \in X$ the function $t \mapsto T(t)x$ is continuous in $[0, +\infty)$ iff it is continuous at 0.

(b) If $\{T(t) : t \geq 0\}$ is strongly continuous, then for any $\delta > 0$ there is $M_\delta > 0$ such that

$$\|T(t)\| \leq M_\delta, \;\; \forall\, t \in [0, \delta].$$

Proof. (a) Using repeatedly the semigroup property in Definition 11.1.1 we get $T(t) = T(\delta)^{n-1}T(t-(n-1)\delta)$ for $(n-1)\delta \le t \le n\delta$, whence $\|T(t)\| \le M^n \le Me^{\omega t}$. Let $x \in X$ be such that $t \mapsto T(t)x$ is continuous at 0, i.e., $\lim_{h\to 0^+} T(h)x = x$. Using again the semigroup property in Definition 11.1.1 it is easily seen that for every $t > 0$ the equality $\lim_{h\to 0^+} T(t+h)x = T(t)x$ holds. Moreover,

$$\|T(t-h)x - T(t)x\| = \|T(t-h)(x - T(h)x)\| \le Me^{\omega(t-h)}\|(x - T(h)x)\|, \qquad 0 < h < t,$$

whence $\lim_{h\to 0^+} T(t-h)x = T(t)x$. It follows that $t \mapsto T(t)x$ is continuous in $[0,+\infty)$.

(b) Let $x \in X$. As $T(\cdot)x$ is continuous, for every $\delta > 0$ there is $M_{\delta,x} > 0$ such that

$$\|T(t)x\| \le M_{\delta,x}, \ \ \forall\, t \in [0,\delta].$$

The statement follows from the Uniform Boundedness Principle, see e.g. [Br, Chapter 2] or [DS1, §II.1]. □

If (11.1.1) holds with $M = 1$ and $\omega = 0$ then the semigroups is said *semigroup of contractions* or *contractive semigroup*. From now on, $\{T(t) : t \ge 0\}$ is a fixed strongly continuous semigroup.

Definition 11.1.3. *The* infinitesimal generator *(or, shortly, the* generator*) of the semigroup $\{T(t) : t \ge 0\}$ is the operator defined by*

$$D(L) = \Big\{x \in X : \exists \lim_{h\to 0^+} \frac{T(h)-I}{h}x\Big\}, \quad Lx = \lim_{h\to 0^+} \frac{T(h)-I}{h}x.$$

By definition, the vector Lx is the right derivative of the function $t \mapsto T(t)x$ at $t = 0$ and $D(L)$ is the subspace where such derivative exists. In general, $D(L)$ is not the whole X, but it is dense, as the next proposition shows.

Proposition 11.1.4. *The domain $D(L)$ of the generator is dense in X.*

Proof. Set

$$M_{a,t}x = \frac{1}{t}\int_a^{a+t} T(s)x\, ds, \ \ a \ge 0, \ t > 0, \ x \in X$$

(this is a X-valued Bochner integral). As the function $s \mapsto T(s)x$ is continuous, we have (see Exercise 11.1)

$$\lim_{t\to 0} M_{a,t}x = T(a)x.$$

In particular, $\lim_{t\to 0^+} M_{0,t}x = x$ for every $x \in X$. Let us show that for every $t > 0$, $M_{0,t}x \in D(L)$, which implies that the statement holds. We have

$$\begin{aligned}\frac{T(h)-I}{h}M_{0,t}x &= \frac{1}{ht}\Big(\int_0^t T(h+s)x\, ds - \int_0^t T(s)x\, ds\Big)\\ &= \frac{1}{ht}\Big(\int_h^{h+t} T(s)x\, ds - \int_0^t T(s)x\, ds\Big)\\ &= \frac{1}{ht}\Big(\int_t^{h+t} T(s)x\, ds - \int_0^h T(s)x\, ds\Big)\\ &= \frac{M_{t,h}x - M_{0,h}x}{t}.\end{aligned}$$

Therefore, for every $x \in X$ we have $M_{0,t}x \in D(L)$ and

$$LM_{0,t}x = \frac{T(t)x - x}{t}. \tag{11.1.2}$$

□

Proposition 11.1.5. *For every $t > 0$, $T(t)$ maps $D(L)$ into itself, and L and $T(t)$ commute on $D(L)$.*

If $x \in D(L)$, then the function $T(\cdot)x$ is differentiable at every $t \geq 0$ and

$$\frac{d}{dt}T(t)x = LT(t)x = T(t)Lx, \;\; t \geq 0.$$

Proof. For every $x \in X$ and for every $h > 0$ we have

$$\frac{T(h) - I}{h}T(t)x = T(t)\frac{T(h) - I}{h}x.$$

If $x \in D(L)$, letting $h \to 0$ we obtain $T(t)x \in D(L)$ and $LT(t)x = T(t)Lx$.

Fix $t_0 \geq 0$ and let $h > 0$. We have

$$\frac{T(t_0 + h)x - T(t_0)x}{h} = T(t_0)\frac{T(h) - I}{h}x \to T(t_0)Lx \quad \text{as } h \to 0^+.$$

This shows that $T(\cdot)x$ is right differentiable at t_0. Let us show that it is left differentiable, assuming $t_0 > 0$. If $h \in (0, t_0)$ we have

$$\frac{T(t_0 - h)x - T(t_0)x}{-h} = T(t_0 - h)\frac{T(h) - I}{h}x \to T(t_0)Lx \quad \text{as } h \to 0^+,$$

as

$$\left\|T(t_0-h)\frac{T(h) - I}{h}x - T(t_0)Lx\right\| \leq \left\|T(t_0-h)\Big(\frac{T(h) - I}{h}x - Lx\Big)\right\| + \|(T(t_0-h) - T(t_0))Lx\|$$

and $\|T(t_0 - h)\| \leq \sup_{0\leq t\leq t_0}\|T(t)\| < \infty$ by Lemma 11.1.2. It follows that the function $t \mapsto T(t)x$ is differentiable at all $t \geq 0$ and its derivative is $T(t)Lx$, which is equal to $LT(t)x$ by the first part of the proof. □

Using Proposition 11.1.5 we prove that the generator L is a closed operator. Therefore, $D(L)$ is a Banach space with the graph norm $\|x\|_{D(L)} = \|x\| + \|Lx\|$.

Proposition 11.1.6. *The generator L of any strongly continuous semigroup is a closed operator.*

Proof. Let (x_n) be a sequence in $D(L)$, and let $x, y \in X$ be such that $x_n \to x$, $Lx_n =: y_n \to y$. By Proposition 11.1.5 the function $t \mapsto T(t)x_n$ is continuously differentiable in $[0, \infty)$. Hence for $0 < h < 1$ we have (see Exercise 11.1)

$$\frac{T(h) - I}{h}x_n = \frac{1}{h}\int_0^h LT(t)x_n dt = \frac{1}{h}\int_0^h T(t)y_n dt,$$

and then

$$\left\|\frac{T(h)-I}{h}x-y\right\| \leq \left\|\frac{T(h)-I}{h}(x-x_n)\right\| + \left\|\frac{1}{h}\int_0^h T(t)(y_n-y)dt\right\| + \left\|\frac{1}{h}\int_0^h T(t)ydt-y\right\|$$
$$\leq \frac{C+1}{h}\|x-x_n\| + C\|y_n-y\| + \left\|\frac{1}{h}\int_0^h T(t)ydt-y\right\|,$$

where $C = \sup_{0<t<1}\|T(t)\|$. Given $\varepsilon > 0$, there is h_0 such that for $0 < h \leq h_0$ we have $\|\int_0^h T(t)ydt/h - y\| \leq \varepsilon/3$. For $h \in (0, h_0]$, take n such that $\|x - x_n\| \leq \varepsilon h/3(C+1)$ and $\|y_n - y\| \leq \varepsilon/3C$: we get $\|\frac{T(h)-I}{h}x - y\| \leq \varepsilon$ and therefore $x \in D(L)$ and $y = Lx$, i.e., the operator L is closed. □

Proposition 11.1.5 implies that for any $x \in D(L)$ the function $u(t) = T(t)x$ is differentiable for $t \geq 0$ and it solves the Cauchy problem

$$\begin{cases} u'(t) = Lu(t), \quad t \geq 0, \\ \\ u(0) = x. \end{cases} \tag{11.1.3}$$

Lemma 11.1.7. *For every $x \in D(L)$, the function $u(t) := T(t)x$ is the* unique *solution of* (11.1.3) *belonging to $C([0,+\infty); D(L)) \cap C^1([0,+\infty); X)$.*

Proof. From Proposition 11.1.5 we know that $u'(t) = T(t)Lx$ for every $t \geq 0$, and then $u' \in C([0,+\infty); X)$. Therefore, $u \in C^1([0,+\infty); X)$. Since $D(L)$ is endowed with the graph norm, a function $u : [0,+\infty) \to D(L)$ is continuous iff both u and Lu are continuous. In our case, both u and $Lu = u'$ belong to $C([0,+\infty); X)$, and then $u \in C([0,+\infty); D(L))$.

Let us prove that (11.1.3) has a unique solution in $C([0,+\infty); D(L)) \cap C^1([0,+\infty); X)$. If $u \in C([0,+\infty); D(L)) \cap C^1([0,+\infty); X)$ is any solution, we fix $t > 0$ and define the function

$$v(s) := T(t-s)u(s), \quad 0 \leq s \leq t.$$

Then (Exercise 11.2) v is differentiable, and $v'(s) = -T(t-s)Lu(s) + T(t-s)u'(s) = 0$ for $0 \leq s \leq t$, whence $v(t) = v(0)$, i.e., $u(t) = T(t)x$. □

Remark 11.1.8. If $\{T(t) : \ t \geq 0\}$ is a C_0-semigroup with generator L, then for every $\lambda \in \mathbb{C}$ the family of operators

$$S(t) = e^{\lambda t}T(t), \quad t \geq 0,$$

is a C_0-semigroup as well, with generator $L + \lambda I : D(L) \to X$. The semigroup property is obvious. Concerning the generator, for every $x \in X$ we have

$$\frac{S(h)x - x}{h} = e^{\lambda h}\frac{T(h) - x}{h} + \frac{e^{\lambda h}x - x}{h}$$

and then

$$\lim_{h\to 0^+}\frac{S(h)x - x}{h} = \lim_{h\to 0^+} e^{\lambda h}\frac{T(h) - x}{h} + \frac{e^{\lambda h}x - x}{h} = Lx + \lambda x$$

iff $x \in D(L)$.

Let $\{T(t) :\ t \geq 0\}$ be a strongly continuous semigroup. Characterising the domain of its generator L may be difficult. However, for many proofs it is enough to know that "good" elements x are dense in $D(L)$. A subspace $D \subset D(L)$ is called a *core* of L if D is dense in $D(L)$ with respect to the graph norm. The following proposition gives an easily checkable sufficient condition in order that D is a core.

Lemma 11.1.9. *If $D \subset D(L)$ is a dense subspace of X and $T(t)(D) \subset D$ for every $t \geq 0$, then D is a core.*

Proof. Let M, ω be such that $\|T(t)\| \leq Me^{\omega t}$ for every $t > 0$. For $x \in D(L)$ we have

$$Lx = \lim_{t \to 0} \frac{1}{t} \int_0^t T(s)Lx\, ds.$$

Let $(x_n) \subset D$ be a sequence such that $\lim_{n\to\infty} x_n = x$. Set

$$y_{n,t} = \frac{1}{t} \int_0^t T(s)x_n\, ds = \frac{1}{t} \int_0^t T(s)(x_n - x)\, ds + \frac{1}{t} \int_0^t T(s)x\, ds.$$

As the $D(L)$-valued function $s \mapsto T(s)x_n$ is continuous in $[0, +\infty)$, the vector $\int_0^t T(s)x_n ds$ belongs to $D(L)$. Moreover, it is the limit of the Riemann sums of elements of D (see Exercise 11.1), hence it belongs to the closure of D in $D(L)$. Therefore, $y_{n,t}$ belongs to the closure of D in $D(L)$ for every n and t. Furthermore,

$$\|y_{n,t} - x\| \leq \left\| \frac{1}{t} \int_0^t T(s)(x_n - x)\, ds \right\| + \left\| \frac{1}{t} \int_0^t T(s)x\, ds - x \right\|$$

tends to 0 as $t \to 0$, $n \to \infty$. By (11.1.2) we have

$$Ly_{n,t} - Lx = \frac{T(t)(x_n - x) - (x_n - x)}{t} + \frac{1}{t} \int_0^t T(s)Lx\, ds - Lx.$$

Given $\varepsilon > 0$, fix $\tau > 0$ such that

$$\left\| \frac{1}{\tau} \int_0^\tau T(s)Lx\, ds - Lx \right\| \leq \varepsilon,$$

and then take $n \in \mathbb{N}$ such that $(Me^{\omega\tau} + 1)\|x_n - x\|/\tau \leq \varepsilon$. Therefore, $\|Ly_{n,\tau} - Lx\| \leq 2\varepsilon$ and the statement follows. □

11.2 Generation Theorems

In this section we recall the main generation theorems for C_0-semigroups. The most general result is the classical Hille–Yosida Theorem, which gives a complete characterisation of the generators. For *contractive* semigroups, i.e., semigroups verifying the estimate $\|T(t)\| \leq 1$ for all $t \geq 0$, the characterisation of the generators provided by the Lumer-Phillips Theorem is often useful. We do not present here the proofs of these results, referring e.g. to [EN, §II.3].

First, we recall the definition of *spectrum* and *resolvent*. The natural setting for spectral theory is that of complex Banach spaces, hence if X is real we replace it by its complexification $\widetilde{X} = \{x + iy : \ x, \ y \in X\}$ endowed with the norm

$$\|x + iy\|_{\tilde{X}} := \sup_{-\pi \le \theta \le \pi} \|x \cos\theta + y \sin\theta\|$$

(notice that the seemingly more natural "Euclidean norm" $(\|x\|^2 + \|y\|^2)^{1/2}$ is not a norm in general).

Definition 11.2.1. *Let $L : D(L) \subset X \to X$ be a linear operator. The* resolvent set $\rho(L)$ *and the* spectrum $\sigma(L)$ *of L are defined by*

$$\rho(L) = \{\lambda \in \mathbb{C} : \exists\, (\lambda I - L)^{-1} \in \mathcal{L}(X)\}, \quad \sigma(L) = \mathbb{C} \backslash \rho(L). \tag{11.2.1}$$

The complex numbers $\lambda \in \sigma(L)$ such that $\lambda I - L$ is not injective are the eigenvalues*, and the vectors $x \in D(L)$ such that $Lx = \lambda x$ are the* eigenvectors *(or* eigenfunctions*, when X is a function space). The set $\sigma_p(L)$ whose elements are all the eigenvalues of L is the* point spectrum*.*

For $\lambda \in \rho(L)$, we set

$$R(\lambda, L) := (\lambda I - L)^{-1}. \tag{11.2.2}$$

The operator $R(\lambda, L)$ is the *resolvent operator* or briefly *resolvent*.

We ask to check (Exercise 11.3) that if the resolvent set $\rho(L)$ is not empty, then L is a closed operator. We also ask to check (Exercise 11.4) the following equality, known as the *resolvent identity*

$$R(\lambda, L) - R(\mu, L) = (\mu - \lambda) R(\lambda, L) R(\mu, L), \ \ \forall\, \lambda, \ \mu \in \rho(L). \tag{11.2.3}$$

Theorem 11.2.2 (Hille–Yosida)**.** *The linear operator $L : D(L) \subset X \to X$ is the generator of a C_0-semigroup verifying estimate* (11.1.1) *iff the following conditions hold:*

$$\begin{cases} (i) & D(L) \ \text{is dense in} \ X, \\[2mm] (ii) & \rho(L) \supset \{\lambda \in \mathbb{R} : \ \lambda > \omega\}, \\[2mm] (iii) & \|(R(\lambda, L))^k\|_{\mathcal{L}(X)} \le \dfrac{M}{(\lambda - \omega)^k} \ \ \forall\, k \in \mathbb{N}, \ \forall\, \lambda > \omega. \end{cases} \tag{11.2.4}$$

Before stating the Lumer–Phillips Theorem, we define the *dissipative* operators.

Definition 11.2.3. *A linear operator $(L, D(L))$ is called* dissipative *if*

$$\|(\lambda I - L)x\| \ge \lambda \|x\|$$

for all $\lambda > 0$, $x \in D(L)$.

Theorem 11.2.4 (Lumer–Phillips)**.** *A densely defined and dissipative operator L on X is closable and its closure is dissipative. Moreover, the following statements are equivalent.*

(i) The closure of L generates a contraction C_0-semigroup.

(ii) The range of $\lambda I - L$ is dense in X for some (hence all) $\lambda > 0$.

11.3 Invariant measures

In our lectures we shall encounter semigroups defined in L^p spaces, i.e., $X = L^p(\Omega)$ where $(\Omega, \mathscr{F}, \mu)$ is a measure space, with $\mu(\Omega) < \infty$. A property that will play an important role is the conservation of the mean value, namely

$$\int_\Omega T(t)f\,d\mu = \int_\Omega f\,d\mu \qquad \forall\, t > 0,\ \forall f \in L^p(\Omega).$$

In this case μ is called *invariant* for $T(t)$. The following proposition gives an equivalent condition for invariance, in terms of the generator of the semigroup rather than the semigroup itself.

Proposition 11.3.1. *Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup with generator L in $L^p(\Omega, \mu)$, where (Ω, μ) is a measure space, $p \in [1, +\infty)$, and $\mu(\Omega) < \infty$. Then*

$$\int_\Omega T(t)f\,d\mu = \int_\Omega f\,d\mu\ \ \forall t > 0,\ \forall f \in L^p(\Omega, \mu) \quad \Longleftrightarrow \int_\Omega Lf\,d\mu = 0\ \ \forall f \in D(L).$$

Proof. "$\Rightarrow$" Let $f \in D(L)$. Then $\lim_{t\to 0}(T(t)f - f)/t = Lf$ in $L^p(\Omega, \mu)$ and consequently in $L^1(\Omega, \mu)$. Integrating we obtain

$$\int_\Omega Lf\,d\mu = \lim_{t\to 0}\frac{1}{t}\int_\Omega (T(t)f - f)d\mu = 0.$$

"$\Leftarrow$" Let $f \in D(L)$. Then the function $t \mapsto T(t)f$ belongs to $C^1([0, +\infty); L^p(\Omega, \mu))$ and $d/dt\,T(t)f = LT(t)f$, so that for every $t \geq 0$,

$$\frac{d}{dt}\int_X T(t)f\,d\mu = \int_\Omega LT(t)f\,d\mu = 0.$$

Therefore the function $t \mapsto \int_X T(t)f\,d\mu$ is constant, and equal to $\int_X f\,d\mu$. The operator $L^p(\Omega, \mu) \to \mathbb{R}$, $f \mapsto \int_\Omega (T(t)f - f)d\mu$, is bounded and vanishes on the dense subset $D(L)$; hence it vanishes in the whole $L^p(\Omega, \mu)$. □

11.4 Analytic semigroups

We recall now an important class of semigroups, the analytic semigroups generated by sectorial operators. For the definition of sectorial operators we need that X is a complex Banach space.

Definition 11.4.1. *A linear operator $L : D(L) \subset X \to X$ is called* sectorial *if there are $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ such that*

$$\begin{cases} (i) \quad \rho(L) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \\ (ii) \quad \|R(\lambda, L)\|_{\mathcal{L}(X)} \leq \dfrac{M}{|\lambda - \omega|}\ \ \forall \lambda \in S_{\theta,\omega}. \end{cases} \tag{11.4.1}$$

Sectorial operators with dense domains are infinitesimal generators of semigroups with noteworthy smoothing properties. The proof of the following theorem may be found in [EN, Chapter 2], [L, Chapter 2].

Theorem 11.4.2. *Let L be a sectorial operator with dense domain. Then it is the infinitesimal generator of a semigroup $\{T(t):\ t\geq 0\}$ that enjoys the following properties.*

(i) $T(t)x \in D(L^k)$ *for every* $t>0$, $x\in X$, $k\in\mathbb{N}$.

(ii) There are $M_0,\ M_1,\ M_2,\ \ldots$, *such that*

$$\begin{cases} (a) \quad \|T(t)\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \ \ t>0, \\ \\ (b) \quad \|t^k (L-\omega I)^k T(t)\|_{\mathcal{L}(X)} \leq M_k e^{\omega t}, \ \ t>0, \end{cases} \tag{11.4.2}$$

where ω *is the constant in* (11.4.1).

(iii) The function $t\mapsto T(t)$ *belongs to* $C^\infty((0,+\infty);\mathcal{L}(X))$, *and the equality*

$$\frac{d^k}{dt^k}T(t) = L^k T(t), \ \ t>0, \tag{11.4.3}$$

holds.

(iv) The function $t\mapsto T(t)$ *has a* $\mathcal{L}(X)$*-valued holomorphic extension in a sector* $S_{\beta,0}$ *with* $\beta>0$.

The name "analytic semigroup" comes from property (iv). If $\mathcal{O}$ is an open set in $\mathbb{C}$, and Y is a complex Banach space, a function $f:\mathcal{O}\to Y$ is called holomorphic if it is differentiable at every $z_0\in\mathcal{O}$ in the usual complex sense, i.e. there exists the limit

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} =: f'(z_0).$$

As in the scalar case, such functions are infinitely many times differentiable at every $z_0\in\mathcal{O}$, and the Taylor series $\sum_{k=0}^{\infty} f^{(k)}(z_0)(z-z_0)^k/k!$ converges to $f(z)$ for every z in a neighborhood of z_0.

We do not present the proof of this theorem, because in the case of Ornstein-Uhlenbeck semigroup that will be discussed in the next lectures we shall provide direct proofs of the relevant properties without relying on the above general results. A more general theory of analytic semigroups, not necessarily strongly continuous at $t=0$, is available, see [L].

11.4.1 Self-adjoint operators in Hilbert spaces

If X is a Hilbert space (inner product $\langle\cdot,\cdot\rangle$, norm $\|\cdot\|$) then we can say more on semigroups and generators in connection to self-adjointness. Notice also that the dissipativity condition can be rephrased in the Hilbert space as follows. An operator $L: D(L)\to X$ is dissipative iff (see Exercise 11.5)

$$\text{Re}\,\langle Lx,x\rangle \leq 0, \ \ \forall x\in D(L). \tag{11.4.4}$$

Let us prove that any self-adjoint dissipative operator is sectorial.

Proposition 11.4.3. *Let $L : D(L) \subset X \to X$ be a self-adjoint dissipative operator. Then L is sectorial with $\theta < \pi$ arbitrary and $\omega = 0$.*

Proof. Let us first show that the spectrum of L is real. If $\lambda = a + ib \in \mathbb{C}$, for every $x \in D(L)$ we have

$$\|(\lambda I - L)x\|^2 = (a^2 + b^2)\|x\|^2 - 2a\langle x, Lx\rangle + \|Lx\|^2 \geq b^2\|x\|^2, \tag{11.4.5}$$

hence if $b \neq 0$ then $\lambda I - L$ is injective. Let us check that in this case it is also surjective, showing that its range is closed and dense in X. Let $(x_n) \subset D(L)$ be a sequence such that the sequence $(\lambda x_n - Lx_n)$ is convergent. From the inequality

$$\|(\lambda I - L)(x_n - x_m)\|^2 \geq b^2\|x_n - x_m\|^2, \;\; n, m \in \mathbb{N},$$

it follows that the sequence (x_n) is a Cauchy sequence, hence (Lx_n) as well. Therefore, there are $x, y \in X$ such that $x_n \to x$ and $Lx_n \to y$. Since L is closed, $x \in D(L)$ and $Lx = y$, hence $\lambda x_n - Lx_n$ converges to $\lambda x - Lx \in \mathrm{rg}\,(\lambda I - L)$ and the range of $\lambda I - L$ is closed.

Let now y be orthogonal to the range of $(\lambda I - L)$. Then, for every $x \in D(L)$ we have $\langle y, \lambda x - Lx\rangle = 0$, whence $y \in D(L^\star) = D(L)$ and $\overline{\lambda} y - L^\star y = \overline{\lambda} y - Ly = 0$. As $\overline{\lambda} I - L$ injective, $y = 0$ follows. Therefore the range of $(\lambda I - L)$ is dense in X.

From the dissipativity of L it follows that the spectrum of L is contained in $(-\infty, 0]$. Indeed, if $\lambda > 0$ then for every $x \in D(L)$ we have, instead of (11.4.5),

$$\|(\lambda I - L)x\|^2 = \lambda^2\|x\|^2 - 2\lambda\langle x, Lx\rangle + \|Lx\|^2 \geq \lambda^2\|x\|^2, \tag{11.4.6}$$

and arguing as above we deeduce $\lambda \in \rho(L)$.

Let us now estimate $\|R(\lambda, L)\|$, for $\lambda = \rho e^{i\theta}$, with $\rho > 0$, $-\pi < \theta < \pi$. For $x \in X$, set $u = R(\lambda, L)x$. Multiplying the equality $\lambda u - Lu = x$ by $e^{-i\theta/2}$ and then taking the inner product with u, we get

$$\rho e^{i\theta/2}\|u\|^2 - e^{-i\theta/2}\langle Lu, u\rangle = e^{-i\theta/2}\langle x, u\rangle,$$

whence, taking the real part,

$$\rho\cos(\theta/2)\|u\|^2 - \cos(\theta/2)\langle Lu, u\rangle = \mathrm{Re}(e^{-i\theta/2}\langle x, u\rangle) \leq \|x\|\,\|u\|$$

and then, as $\cos(\theta/2) > 0$, also

$$\|u\| \leq \frac{\|x\|}{|\lambda|\cos(\theta/2)},$$

with $\theta = \arg\lambda$. □

Proposition 11.4.4. *Let $\{T(t) : \;\; t \geq 0\}$ be a C_0-semigroup. The family of operators $\{T^\star(t) : \;\; t \geq 0\}$ is a C_0-semigroup whose generator is $L^\star$.*

Proof. The semigroup law is immediately checked. Let us prove the strong continuity. Possibly considering the rescaled semigroup $e^{-\omega t}T(t)$ with M, ω as in (11.1.1), see Remark 11.1.8, we may assume that $\|T(t)\|_{\mathcal{L}(X)} \leq M$ for every $t \geq 0$, without loss of generality, $\|T(t)\| = \|T(t)^\star\| \leq 1$ (see Exercise 11.6). For $x \in X$ we have

$$\begin{aligned}\|T(t)^\star x - x\|^2 &= \langle T(t)^\star x - x, T(t)^\star x - x\rangle \\ &= \|T(t)^\star x\|^2 + \|x\|^2 - \langle x, T(t)^\star x\rangle - \langle T(t)^\star x, x\rangle \\ &\leq 2\|x\|^2 - \big(\langle x, T(t)^\star x\rangle + \langle T(t)^\star x, x\rangle\big) \\ &= 2\|x\|^2 - \big(\langle T(t)x, x\rangle + \langle x, T(t)x\rangle\big)\end{aligned}$$

whence

$$\limsup_{t\to 0} \|T(t)^\star x - x\| = 0$$

by the strong continuity of $T(t)$, and then $T(\cdot)^\star x$ is continuous at 0. By Lemma 11.1.2, $t \mapsto T(t)^\star x$ is continuous on $[0, \infty)$ and $\{T(t)^\star : \ t \geq 0\}$ is a C_0-semigroup. Denoting by A its generator, for $x \in D(L)$ and $y \in D(A)$ we have

$$\langle Lx, y\rangle = \lim_{t\to 0}\langle t^{-1}(T(t) - I)x, y\rangle = \lim_{t\to 0}\langle x, t^{-1}(T(t)^\star - I)y\rangle = \langle x, Ay\rangle,$$

so that $A \subset L^\star$. Conversely, for $y \in D(L^\star), \ x \in D(L)$ we have

$$\begin{aligned}\langle x, T(t)^\star y - y\rangle = \langle T(t)x - x, y\rangle &= \int_0^t \langle LT(s)x, y\rangle \, ds \\ &= \int_0^t \langle T(s)x, L^\star y\rangle \, ds = \int_0^t \langle x, T(s)^\star L^\star y\rangle \, ds.\end{aligned}$$

We deduce

$$T(t)^\star y - y = \int_0^t T(s)^\star L^\star y \, ds,$$

whence, dividing by t and letting $t \to 0$ we get $Ay = L^\star y$ for every $y \in D(L^\star)$ and consequently $L^\star \subset A$. □

The following result is an immediate consequence of Proposition 11.4.4.

Corollary 11.4.5. *The generator L is self-adjoint if and only if $T(t)$ is self-adjoint for every $t > 0$.*

11.5 Exercises

Exercise 11.1. Let $\mathbb{R}$ be endowed with the Lebesgue measure λ_1, and let $f : [a, b] \to X$ be a continuous function. Prove that it is Bochner integrable, that

$$\int_a^b f(t) \, dt = \lim_{n\to\infty} \sum_{i=1}^n f(\tau_i) \frac{b-a}{n}$$

for any choice of $\tau_i \in \left[a + \frac{(b-a)(i-1)}{n}, a + \frac{(b-a)i}{n}\right]$, $i = 1, \dots, n$ (the sums in this approximation are the usual Riemann sums in the real-valued case) and that, setting

$$F(t) = \int_a^t f(s)ds, \quad a \le t \le b,$$

the function F is continuously differentiable, with

$$F'(t) = f(t), \quad a \le t \le b.$$

Exercise 11.2. Prove that if $u \in C([0,+\infty); D(L)) \cap C^1([0,+\infty); X)$ is a solution of problem (11.1.3), then for $t > 0$ the function $v(s) = T(t-s)u(s)$ is continuously differentiable in $[0,t]$ and it verifies $v'(s) = -T(t-s)Lu(s) + T(t-s)u'(s) = 0$ for $0 \le s \le t$.

Exercise 11.3. Let $L : D(L) \subset X \to X$ be a linear operator. Prove that if $\rho(L) \neq \emptyset$ then L is closed.

Exercise 11.4. Prove the resolvent identity (11.2.3).

Exercise 11.5. Prove that in Hilbert spaces the dissipativity condition in Definition 11.2.3 is equivalent to (11.4.4).

Exercise 11.6. Let $\{T(t) : \ t \ge 0\}$ be a bounded strongly continuous semigroup. Prove that the norm

$$|x| := \sup_{t \ge 0} \|T(t)x\|$$

is equivalent to $\|\cdot\|$ and that $T(t)$ is contractive on $(X, |\cdot|)$.

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