Proposition 11.4.4. *Let $\{T(t):\ t\ge 0\}$ be a C_0-semigroup. The family of operators $\{T(t)^\star:\ t\ge 0\}$ is a C_0-semigroup whose generator is $L^\star$.*

Proof. The semigroup law is immediately checked. Let us prove the strong continuity. Recall that by (11.1.1) we have $\|T(t)\| = \|T(t)^\star\| \le Me^{\omega t}$, where we may assume $\omega > 0$. First, notice that $T(t)^\star x \to x$ weakly for every $x \in X$ as $t \to 0$. Indeed, by the strong continuity of $T(t)$ we have $\langle T(t)^\star x, y\rangle = \langle x, T(t)y\rangle \to \langle x, y\rangle$ as $t \to 0$ for every $y \in X$. From the estimate

$$\Big|\int_0^t \langle T(s)^\star x, y\rangle\, ds\Big| \le \frac{M}{\omega}(e^{\omega t}-1)\|x\|\,\|y\|$$

and the Riesz Theorem we get the existence of $x_t \in X$ such that

$$\frac{1}{t}\int_0^t \langle T(s)^\star x, y\rangle\, ds = \langle x_t, y\rangle \qquad \forall\ y \in X.$$

Therefore, for $t > 0$ and $0 < h < t$ we infer

$$\begin{aligned}
|\langle T(h)^\star x_t - x_t, y\rangle| &= |\langle x_t, T(h)y\rangle - \langle x_t, y\rangle| \\
&= \Big|\frac{1}{t}\int_0^t \langle T(s)^\star x, T(h)y\rangle\, ds - \frac{1}{t}\int_0^t \langle T(s)^\star x, y\rangle\, ds\Big| \\
&= \Big|\frac{1}{t}\int_0^t \langle T(s+h)^\star x, y\rangle\, ds - \frac{1}{t}\int_0^t \langle T(s)^\star x, y\rangle\, ds\Big| \\
&= \Big|\frac{1}{t}\int_t^{t+h} \langle T(s)^\star x, y\rangle\, ds - \frac{1}{t}\int_0^h \langle T(s)^\star x, y\rangle\, ds\Big| \\
&\le \frac{1}{t}\|x\|\,\|y\|\,\frac{M}{\omega}\big[(e^{\omega(t+h)} - e^{\omega t}) + (e^{\omega h}-1)\big].
\end{aligned}$$

Taking the supremum on $\|y\| = 1$, we deduce

$$\lim_{h\to 0}\|T(h)^\star x_t - x_t\| = 0. \tag{11.4.7}$$

Since by linearity and an $\varepsilon/3$ argument the set where (11.4.7) holds is a weakly closed subspace Y, containing x_t for all $t > 0$. Since for any $x \in X$, $x_t \to x$ weakly as $t \to 0$, we conclude that $Y = X$ and that $T(t)^\star$ is strongly continuous. Denoting by A its generator, for $x \in D(L)$ and $y \in D(A)$ we have

$$\langle Lx, y\rangle = \lim_{t\to 0}\langle t^{-1}(T(t)-I)x, y\rangle = \lim_{t\to 0}\langle x, t^{-1}(T(t)^\star - I)y\rangle = \langle x, Ay\rangle,$$

so that $A \subset L^\star$. Conversely, for $y \in D(L^\star),\ x \in D(L)$ we have

$$\begin{aligned}
\langle x, T(t)^\star y - y\rangle = \langle T(t)x - x, y\rangle &= \int_0^t \langle LT(s)x, y\rangle\, ds \\
&= \int_0^t \langle T(s)x, L^\star y\rangle\, ds = \int_0^t \langle x, T(s)^\star L^\star y\rangle\, ds.
\end{aligned}$$

We deduce

$$T(t)^\star y - y = \int_0^t T(s)^\star L^\star y \, ds,$$

whence, dividing by t and letting $t \to 0$ we get $Ay = L^\star y$ for every $y \in D(L^\star)$ and consequently $L^\star \subset A$. $\square$