# Solution to the Exercises of Lecture 10

Team Salerno

**Exercise 10.1:** Prove that for $f \in \mathcal{F}C_b^1(X)$

$$\nabla_H \mathbb{E}_n f(x) = \int_X P_n \nabla_H f(P_n x + (I - P_n)y)\gamma(dy), \quad \forall x \in X,$$

holds.

PROOF: Let us recall, by (7.4.1), that

$$\mathbb{E}_n f(x) = \int_X f(P_n x + (I - P_n)y)\,\gamma(dy), \quad x \in X,$$

where $P_n x = \sum_{j=1}^n \hat{h}_j(x) h_j$, $n \in \mathbb{N}$, $x \in X$, with $h_j = R_\gamma \hat{h}_j$ an orthonormal basis of the Cameron-Martin space $H$.
So, by the dominated convergence theorem, one obtains

$$\partial_j \mathbb{E}_n f(x) = \int_X \frac{\partial f}{\partial h_j}(P_n x + (I - P_n)y)\gamma(dy)$$

for $j \leq n$. Since for every $y \in X$ the directional derivatives along all $h_j$ of the function $f(P_n \cdot + (I - P_n)y)$ vanish for $j > n$, it follows from (9.3.2) that

$$\begin{aligned}
\nabla_H \mathbb{E}_n f(x) &= \sum_{j=1}^{\infty} \partial_j \mathbb{E}_n f(x) h_j \\
&= \sum_{j=1}^{n} \int_X \frac{\partial f}{\partial h_j}(P_n x + (I - P_n)y) h_j \gamma(dy) \\
&= \int_X \sum_{j=1}^{n} \frac{\partial f}{\partial h_j}(P_n x + (I - P_n)y) h_j \gamma(dy) \\
&= \int_X P_n \nabla_H f(P_n x + (I - P_n)y)\gamma(dy), \quad x \in X.
\end{aligned}$$

**Exercise 10.2:** Prove that if $f \in \mathcal{F}C^1(X) \cap L^p(X, \gamma)$, $1 \leq p < \infty$ and $\nabla_H f \in L^p(X, \gamma, H)$ then $f \in W^{1,p}(X, \gamma)$.

PROOF: There exists $\varphi \in C^1(\mathbb{R}^n)$, $n \geq 1$, $l_1, \ldots, l_n \in X^*$ such that

$$f(x) = \varphi(l_1(x), \ldots, l_n(x)), \quad x \in X.$$

Consider $\eta \in C_c^1(\mathbb{R}^n)$ such that $\chi_{B(1)} \leq \eta \leq \chi_{B(2)}$ where $B(1)$ and $B(2)$ are the centred open balls of radius 1 and 2 respectively. Now, for $m \geq 1$ and $y \in \mathbb{R}^n$, we set $\eta_m(y) = \eta(\frac{y}{m})$ and $\varphi_m(y) = \eta_m(y)\varphi(y)$. It is obvious that $\varphi_m \in C_b^1(\mathbb{R}^n)$, and hence

$$f_m(\cdot) = \varphi_m(l_1(\cdot), \ldots, l_n(\cdot)) \in \mathcal{F}C_b^1(X).$$

Since $\lim_{m\to\infty} \eta_m(y) = 1$ and $0 \leq \eta_m(y) \leq 1$, it follows that for every $x \in X$,

$$\begin{aligned} \lim_{m\to\infty} f_m(x) &= f(x) \text{ and} \\ |f_m(x)| &\leq |f(x)|. \end{aligned}$$

So, by the dominated convergence theorem one obtains $f_m \longrightarrow f$ in $L^p(X, \gamma)$. On the other hand, for $v \in X$, we have

$$\begin{aligned} f_m'(x)(v) &= \langle \nabla\varphi_m(l(x)), l(v)\rangle_{\mathbb{R}^n} \\ &= \frac{\varphi(l(x))}{m}\langle \nabla\eta(l(x)/m), l(v)\rangle_{\mathbb{R}^n} + \eta_m(l(x))\langle\nabla\varphi(l(x)), l(v)\rangle_{\mathbb{R}^n} \\ &= \frac{f(x)}{m}\langle \nabla\eta(l(x)/m), l(v)\rangle_{\mathbb{R}^n} + \eta_m(l(x))f'(x)(v), \end{aligned}$$

where $l(v) = (l_1(v), \ldots, l_n(v))$. Thus, by Lemma 9.3.4, we have

$$\begin{aligned} & |\nabla_H f_m(x) - \nabla_H f(x)|_H \\ = \; & |f_m'(x)(\cdot) - f'(x)(\cdot)|_{L^2(X,\gamma)} \\ = \; & \left| (\eta_m(l(x)) - 1)f'(x)(\cdot) + \frac{f(x)}{m}\langle\nabla\eta(l(x)/m), l(\cdot)\rangle_{\mathbb{R}^n} \right|_{L^2(X,\gamma)} \\ \leq \; & |\eta_m(l(x)) - 1||f'(x)|_{L^2(X,\gamma)} + \frac{M\|\nabla\eta\|_\infty}{m}|f(x)|, \end{aligned}$$

where $M^2 = \int_{\mathbb{R}^n} |y|^2(\gamma \circ l^{-1})(dy) < \infty$. Hence, $\nabla_H f_m(x) \longrightarrow \nabla_H f(x)$ in $H$. Moreover

$$\begin{aligned} |\nabla_H f_m(x)|_H = |f_m'(x)|_{L^2(X,\gamma)} &\leq \eta_m(l(x))|f'(x)|_{L^2(X,\gamma)} + \frac{M\|\nabla\eta\|_\infty}{m}|f(x)| \\ &\leq |\nabla_H f(x)|_H + M\|\nabla\eta\|_\infty|f(x)| \end{aligned}$$

Applying the dominated convergence theorem one obtains that $\nabla_H f_m$ converges to $\nabla_H f$ in $L^p(X, \gamma, H)$. Thus, $f \in W^{1,p}(X, \gamma)$.

**Exercise 10.3:** Prove that if $f \in W^{1,p}(X, \gamma)$ then $f^+$, $f^-$, $|f| \in W^{1,p}(X, \gamma)$ as well. Compute $\nabla_H f^+$, $\nabla_H f^-$, $\nabla_H |f|$ and deduce that $\nabla_H f = 0$ a.e. on

$\{f = c\}$ for every $c \in \mathbb{R}$.

Proof: Let us recall first that $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$. Consider the functions

$$\theta_\epsilon(s) = \begin{cases} \sqrt{s^2 + \epsilon^2} - \epsilon & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases}$$

where $\epsilon > 0$, and $\eta \in C_c^\infty(\mathbb{R})$ with $0 \le \eta \le 1$, $\eta(s) = 1$ for $|s| \le 1$, and $\eta(s) = 0$ if $|s| \ge 2$.
So, the function $\psi_n(s) := \eta_n(s)\theta_\epsilon(s)$ belongs to $C_b^1(\mathbb{R})$, where $\eta_n(s) := \eta(s/n)$. Applying Proposition 9.3.10.(ii), we deduce that $\psi_n \circ f \in W^{1,p}(X, \gamma)$, and

$$\nabla_H(\psi_n \circ f) = \psi_n'(f)\nabla_H f = (\eta_n'(f)\theta_\epsilon(f) + \eta_n(f)\theta_\epsilon'(f))\nabla_H f.$$

Thus,

$$\lim_{n\to\infty} (\psi_n \circ f)(x) = (\theta_\epsilon \circ f)(x), \ \text{ and } \lim_{n\to\infty} \nabla_H(\psi_n \circ f)(x) = \theta_\epsilon'(f(x))\nabla_H f(x)$$

for $\gamma$-a.e. $x \in X$. Moreover, $|\psi_n \circ f| \le |\theta_\epsilon \circ f| \le |f|$, and

$$\begin{aligned} |\nabla_H(\psi_n \circ f)|_H &\le \frac{\|\eta'\|_\infty}{n}\chi_{[n<|f|<2n]}\theta_\epsilon(f)|\nabla_H f|_H + \theta_\epsilon'(f)|\nabla_H f|_H \\ &\le (2\|\eta'\|_\infty + 1)|\nabla_H f|_H. \end{aligned}$$

Hence, by the dominated convergence theorem, we have $\theta_\epsilon \circ f \in W^{1,p}(X, \gamma)$ and

$$\nabla_H(\theta_\epsilon \circ f) = \theta_\epsilon'(f)\nabla_H f = \begin{cases} \frac{f}{\sqrt{f^2+\epsilon^2}}\nabla_H f & \text{if } f > 0, \\ 0 & \text{if } f \le 0. \end{cases}$$

Since $\lim_{\epsilon\to 0}\theta_\epsilon(f) = f^+$ and $\lim_{\epsilon\to 0}\nabla_H(\theta_\epsilon \circ f) = \begin{cases} \nabla_H f & \text{if } f > 0, \\ 0 & \text{if } f \le 0. \end{cases}$ $\gamma$-a.e., and $|\theta_\epsilon \circ f| \le |f|$, $|\nabla_H(\theta_\epsilon \circ f)|_H \le |\nabla_H f|_H$, it follows from the dominated convergence theorem that $f^+ \in W^{1,p}(X, \gamma)$ and

$$\nabla_H f^+ = \begin{cases} \nabla_H f & \text{if } f > 0, \\ 0 & \text{if } f \le 0. \end{cases}$$

Now, by noting that $f^- = (-f)^+$ and $|f| = f^+ + f^-$ we deduce that $f^-$, $|f| \in W^{1,p}(X, \gamma)$ and

$$\nabla_H f^- = \begin{cases} -\nabla_H f & \text{if } f < 0, \\ 0 & \text{if } f \ge 0, \end{cases}$$

$$\nabla_H |f| = \begin{cases} \nabla_H f & \text{if } f > 0, \\ 0 & \text{if } f = 0, \\ -\nabla_H f & \text{if } f < 0. \end{cases}$$

We prove now that $\nabla_H f = 0$ a.e. on $\{f = c\}$ for every $c \in \mathbb{R}$. Without loss of generality one can assume that $c = 0$. The claim follows from the expressions of $\nabla_H f^+$ and $\nabla_H f^-$ since $\nabla_H f = \nabla_H f^+ - \nabla_H f^-$.

**Exercise 10.4:** Let $\varphi \in W^{1,p}(\mathbb{R}^n, \gamma_n)$ and let $l_1, \ldots, l_n \in X^*$, with $\langle l_i, l_j \rangle_{L^2(X,\gamma)} = \delta_{ij}$. Prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = \varphi(l_1(x), \ldots, l_n(x))$ belongs to $W^{1,p}(X, \gamma)$.

PROOF: Since $\varphi \in W^{1,p}(\mathbb{R}^N, \gamma_n)$, by Definition 9.1.4, there is $(\varphi_m) \subset C_b^1(\mathbb{R}^n)$ such that $\varphi_m \to \varphi$ in $L^p(\mathbb{R}^n, \gamma_n)$ and $\nabla \varphi_m \to \nabla \varphi$ in $L^p(\mathbb{R}^n, \gamma_n, \mathbb{R}^n)$. Define the function $f_m \in \mathcal{F}C_b^1(X)$ by

$$f_m(x) = \varphi_m(T_n(x)), \quad x \in X,$$

where $T_n(x) = (l_1(x), \ldots, l_n(x))$. Using $\gamma_n = \gamma \circ T_n^{-1}$, see Exercise 2.4, we have

$$\int_X |f(x) - f_m(x)|^p \, \gamma(dx) = \int_{\mathbb{R}^n} |\varphi(z) - \varphi_m(z)|^p \, \gamma_n(dz).$$

This implies that $f_m \to f$ in $L^p(X, \gamma)$. On the other hand, since $\langle l_i, l_j \rangle_{L^2(X,\gamma)} = \delta_{ij}$, we have

$$|\nabla_H f_m(x)|_H^2 = |\nabla \varphi_m(T_n(x))|_{\mathbb{R}^n}^2.$$

Thus,

$$\begin{aligned}
& \int_X |\nabla_H f_m(x) - \nabla_H f_{m'}(x)|_H^p \gamma(dx) \\
= & \int_X |\nabla \varphi_m(T_n(x)) - \nabla \varphi_{m'}(T_n(x))|_{\mathbb{R}^n}^p \gamma(dx) \\
= & \int_{\mathbb{R}^n} |\nabla \varphi_m(z) - \nabla \varphi_{m'}(z)|_{\mathbb{R}^n}^p \gamma_n(dx),
\end{aligned}$$

where we apply again that $\gamma_n = \gamma \circ T_n^{-1}$. So, $(f_m)$ is a Cauchy sequence in $W^{1,p}(X, \gamma)$. Hence, there is $g \in W^{1,p}(X, \gamma)$ such that $f_m \to g$ in $W^{1,p}(X, \gamma)$. Since $f_m \to f$ in $L^p(X, \gamma)$, we deduce that $f = g$.

**Exercise 10.5:** Let $f \in L^p(X, \gamma)$, $p > 1$, be such that $\mathbb{E}_n f \in W^{1,p}(X, \gamma)$ for every $n \in \mathbb{N}$, with $\sup_n \|\nabla_H \mathbb{E}_n f\|_{L^p(X,\gamma;H)} < \infty$. Prove that $f \in W^{1,p}(X, \gamma)$.

PROOF: If follows from the assumption $\sup_n \|\nabla_H \mathbb{E}_n f\|_{L^p(X,\gamma;H)} < \infty$ and (7.4.2) that the sequence $(\mathbb{E}_n f)$ is bounded in $W^{1,p}(X, \gamma)$. So, by reflexivity, there is a subsequence $(\mathbb{E}_{n_k} f)$ converging weakly to some $g \in W^{1,p}(X, \gamma)$ as $k \to \infty$. Since, by Proposition 7.4.5, $\mathbb{E}_{n_k} f \to f$ in $L^p(X, \gamma)$, it follows that $g = f$. Hence $f \in W^{1,p}(X, \gamma)$.

**Exercise 10.6:** Prove that $\mathcal{F}C_b^2(X)$ is dense in $W^{1,2}(X, \gamma)$.

PROOF: By the definition of $W^{1,2}(X,\gamma)$ we have only to prove that $\mathcal{F}C_b^2(X)$ is dense in $\mathcal{F}C_b^1(X)$ with respect to the $W^{1,2}(X,\gamma)$-norm. Let $f \in \mathcal{F}C_b^1(X)$. So, $f(\cdot) = \varphi(l_1(\cdot), \ldots, l_n(\cdot))$ for some $\varphi \in C_b^1(\mathbb{R}^n)$ and $l_1, \ldots, l_n \in X^*$. Let $(\rho_m)$ be a sequence of mollifiers and $\varphi_m = \rho_m * \varphi$, $m \in \mathbb{N}$. It is well known that $\rho_m \in C_c^\infty(\mathbb{R}^n)$, which implies, in particular, that $\varphi_m \in C_b^2(\mathbb{R}^n)$ and hence $f_m = \varphi_m(l_1(\cdot), \ldots, l_n(\cdot)) \in \mathcal{F}C_b^2(X)$. Moreover, for any $y \in \mathbb{R}^n$,

$$|\varphi_m(y)| = \left| \int_{\mathbb{R}^n} \rho_m(z)\varphi(y-z)dz \right| \leq \|\varphi\|_\infty \int_{\mathbb{R}^n} \rho_m(z)dz = \|\varphi\|_\infty, \quad (1)$$

$$|\nabla\varphi_m(y)| = \left| \int_{\mathbb{R}^n} \rho_m(z)\nabla\varphi(y-z)dz \right| \leq \|\nabla\varphi\|_\infty. \quad (2)$$

On the other hand, we know that $\lim_{m\to\infty} \varphi_m(y) = \varphi(y)$ for any $y \in \mathbb{R}^n$, and hence,

$$\lim_{m\to\infty} f_m(x) = f(x), \qquad x \in X.$$

Taking into account the estimate (1), one can apply the dominated convergence theorem to get that $f_m$ converges to $f$ in $L^2(X,\gamma)$.
Similarly, since $\lim_{m\to\infty} \nabla\varphi_m(y) = \nabla\varphi(y)$, $y \in \mathbb{R}^n$, as in Exercise 10.2 one has, for $x \in X$,

$$\begin{aligned} |\nabla_H f_m(x) - \nabla_H f(x)|_H &= |f_m'(x)(\cdot) - f'(x)(\cdot)|_{L^2(X,\gamma)} \\ &\leq M|\nabla\varphi_m(l(x)) - \nabla\varphi(l(x))|_{\mathbb{R}^n}, \end{aligned}$$

where $M$ is as in Exercise 10.2. Hence, $\nabla_H f_m(x) \longrightarrow \nabla_H f(x)$ in $H$. Using now (2) one obtains, by the dominated convergence theorem, that $\nabla_H f_m$ converges to $\nabla_H f$ in $L^2(X,\gamma,H)$. This ends the proof.