Lecture 10

Sobolev Spaces II

In this Lecture we go on in the description of the Sobolev spaces $W^{1,p}(X,\gamma)$, and we define the Sobolev spaces $W^{2,p}(X,\gamma)$. We give approximation results through the cylindrical functions $\mathbb{E}_n f$, and we introduce the divergence of vector fields; formally, the divergence operator is adjoint of the H-gradient. We use the notation of Lecture 9. So, X is a separable Banach space endowed with a centred nondegenerate Gaussian measure γ, and if $\{h_j:\ j\in\mathbb{N}\}\subset R_\gamma(X^*)$ is an orthonormal basis of the Cameron-Martin space H, then for every $f\in W^{1,p}(X,\gamma)$ we denote by $\partial_j f(x)=[\nabla_H f(x),h_j]_H$ the generalised derivative of f in the direction h_j.

10.1 Further properties of $W^{1,p}$ spaces

Let $f\in W^{1,p}(X,\gamma)$. For every $h\in H$, $\partial_h f$ plays the role of weak derivative of f in the h direction. Indeed, by Proposition 9.3.10, for every $f\in W^{1,p}(X,\gamma)$ and $\varphi\in C^1_b(X)$, applying formula (9.3.3) to the product $f\varphi$ we get

$$\int_X (\partial_h\varphi) f\,d\gamma = -\int_X \varphi(\partial_h f)\,d\gamma + \int_X \varphi\, f\,\hat h\,d\gamma.$$

The Sobolev spaces may be defined through the weak derivatives. Given $f\in L^p(X,\gamma)$ and $h\in H$, a function $g\in L^1(X,\gamma)$ is called *weak derivative* of f in the direction of h if

$$\int_X (\partial_h\varphi) f\,d\gamma = -\int_X \varphi\, g\,d\gamma + \int_X \varphi\, f\,\hat h\,d\gamma,\quad \forall\varphi\in C^1_b(X).$$

The weak derivative is unique, because if $\int_X \varphi\, g\,d\gamma=0$ for every $\varphi\in C^1_b(X)$, then $g=0$ a.e. by Lemma 9.3.6.

We set

$$G^{1,p}(X,\gamma)=\Big\{f\in L^p(X,\gamma):\exists\Psi\in L^p(X,\gamma;H)\text{ such that for each }h\in H,\ [\Psi(\cdot),h]_H\text{ is the weak derivative of }f\text{ in the direction }h\Big\}.$$

If $f \in G^{1,p}(X,\gamma)$ and Ψ is the function in the definition, we set

$$D_H f := \Psi, \quad \|f\|_{G^{1,p}(X,\gamma)} = \|f\|_{L^p(X,\gamma)} + \|\Psi\|_{L^p(X,\gamma;H)}.$$

Theorem 10.1.1. *For every $p > 1$, $G^{1,p}(X,\gamma) = W^{1,p}(X,\gamma)$ and $D_H f = \nabla_H f$ for every $f \in W^{1,p}(X,\gamma)$.*

The proof may be found e.g. in [B, Cor. 5.4.7].

Let us come back to the approximation by conditional expectations introduced in Subsection 7.4. We already know that if $f \in L^p(X,\gamma)$ then $\mathbb{E}_n f \to f$ in $L^p(X,\gamma)$ as $n \to \infty$.

Proposition 10.1.2. *Let $1 \le p < \infty$ and let $f \in W^{1,p}(X,\gamma)$. Then, $\mathbb{E}_n f \in W^{1,p}(X,\gamma)$ for all $n \in \mathbb{N}$ and:*

(i) *for every $j \in \mathbb{N}$*

$$\partial_j(\mathbb{E}_n f) = \begin{cases} \mathbb{E}_n(\partial_j f) & \text{if } j \le n, \\ 0 & \text{if } j > n; \end{cases} \tag{10.1.1}$$

(ii) $\|\mathbb{E}_n f\|_{W^{1,p}(X,\gamma)} \le \|f\|_{W^{1,p}(X,\gamma)}$*;*

(iii) $\lim_{n\to\infty} \mathbb{E}_n f = f$ *in $W^{1,p}(X,\gamma)$.*

Proof. Let $f \in \mathcal{F}C^1_b(X)$. Since $P_n x = \sum_{i=1}^n \hat{h}_i(x) h_i$ and $\frac{\partial \hat{h}_i}{\partial h_j}(x) = \delta_{ij}$ for every x, for every $y \in X$ the function $x \mapsto f(P_n x + (I - P_n)y)$ has directional derivatives along all h_j, that vanish for $j > n$ and are equal to $\frac{\partial f}{\partial h_j}(P_n x + (I - P_n)y)$ for $j \le n$.
Since $x \mapsto \frac{\partial f}{\partial h_j}(P_n x + (I - P_n)y)$ is continuous and bounded by a constant independent of y, for $j \le n$ we get

$$\partial_j \mathbb{E}_n f(x) = \frac{\partial}{\partial h_j}\int_X f(P_n x + (I - P_n)y)\gamma(dy) = \int_X \frac{\partial f}{\partial h_j}(P_n x + (I - P_n)y)\gamma(dy).$$

In other words, (i) holds, and it yields

$$\nabla_H \mathbb{E}_n f(x) = \int_X P_n \nabla_H f(P_n x + (I - P_n)y)\gamma(dy), \qquad \forall x \in X. \tag{10.1.2}$$

So we have

$$\begin{aligned}\|\nabla_H \mathbb{E}_n f - \nabla_H f\|^p_{L^p(X,\gamma;H)} &= \int_X \left| \int_X (P_n \nabla_H f(P_n x + (I - P_n)y) - \nabla_H f(x))\gamma(dy)\right|_H^p \gamma(dx) \\ &\le \int_X \int_X |P_n \nabla_H f(P_n x + (I - P_n)y) - \nabla_H f(x)|_H^p \,\gamma(dy)\gamma(dx).\end{aligned}$$

Notice that

$$\lim_{n\to+\infty} |P_n \nabla_H f(P_n x + (I - P_n)y) - \nabla_H f(x)|_H = 0, \qquad \gamma\otimes\gamma - a.e.\ (x,y).$$

Indeed, recalling that $\|P_n\|_{\mathcal{L}(H)} \leq 1$,

$$\begin{aligned}
&|P_n\nabla_H f(P_nx + (I-P_n)y) - \nabla_H f(x)|_H\\
&\leq \Big|P_n\Big(\nabla_H f(P_nx + (I-P_n)y) - \nabla_H f(x)\Big)\Big|_H + |P_n\nabla_H f(x) - \nabla_H f(x)|_H\\
&\leq |\nabla_H f(P_nx + (I-P_n)y) - \nabla_H f(x)|_H + |(P_n - I)\nabla_H f(x)|_H
\end{aligned}$$

and the first addendum vanishes as $n \to +\infty$ for $\gamma \otimes \gamma$–a.e. (x, y) since

$$\lim_{n\to+\infty} P_nx + (I-P_n)y = x$$

for $\gamma \otimes \gamma$-a.e. (x, y) and $\nabla_H f$ is continuous; the second addendum goes to 0 as $n \to \infty$ for every $x \in X$. Moreover,

$$|P_n\nabla_H f(P_nx + (I-P_n)y) - \nabla_H f(x)|_H \leq 2\sup_{z\in X} |\nabla_H f(z)|_H.$$

By the Dominated Convergence Theorem,

$$\lim_{n\to+\infty} \nabla_H \mathbb{E}_n f = \nabla_H f$$

in $L^p(X, \gamma; H)$, and taking into account Proposition 7.4.5,

$$\lim_{n\to+\infty} \mathbb{E}_n f = f$$

in $W^{1,p}(X, \gamma)$. So, f satisfies (iii). Moreover,

$$\begin{aligned}
\|\nabla_H \mathbb{E}_n f\|^p_{L^p(X,\gamma;H)} &= \int_X \left|\int_X P_n\nabla_H f(P_nx + (I-P_n)y)\gamma(dy)\right|^p_H \gamma(dx)\\
&\leq \int_X \int_X |P_n\nabla_H f(P_nx + (I-P_n)y)|^p_H\, \gamma(dy)\gamma(dx)\\
&\leq \int_X \int_X |\nabla_H f(P_nx + (I-P_n)y)|^p_H \gamma(dy)\gamma(dx)\\
&= \int_X |\nabla_H f(x)|^p_H \gamma(dx)
\end{aligned} \tag{10.1.3}$$

where the last equality follows from Proposition 7.3.2.

Estimate (10.1.3) and (7.4.2) yield (ii) for $f \in \mathcal{F}C^1_b(X)$.

Let now $f \in W^{1,p}(X, \gamma)$, and let $(f_k) \subset \mathcal{F}C^1_b(X)$ be a sequence converging to f in $W^{1,p}(X, \gamma)$. By estimate (7.4.2), for every $n \in \mathbb{N}$ the sequence $(\mathbb{E}_n f_k)_k$ converges to $\mathbb{E}_n f$ in $L^p(X, \gamma)$, and by (ii) $(\mathbb{E}_n f_k)_k$ is a Cauchy sequence in $W^{1,p}(X, \gamma)$. Therefore, $\mathbb{E}_n f \in W^{1,p}(X, \gamma)$ and

$$\nabla_H \mathbb{E}_n f = \lim_{k\to+\infty} \nabla_H \mathbb{E}_n f_k$$

in $L^p(X, \gamma; H)$ so that

$$\begin{aligned}
\|\nabla_H \mathbb{E}_n f\|_{L^p(X,\gamma;H)} =& \lim_{k\to+\infty} \|\nabla_H \mathbb{E}_n f_k\|_{L^p(X,\gamma;H)} \leq \lim_{k\to+\infty} \|\nabla_H f_k\|_{L^p(X,\gamma;H)}\\
=&\|\nabla_H f\|_{L^p(X,\gamma;H)}.
\end{aligned}$$

Therefore (ii) holds for every $f \in W^{1,p}(X,\gamma)$ and then (iii) follows from (ii) and from the density of $\mathcal{F}C^1_b(X)$ in $W^{1,p}(X,\gamma)$.
(i) follows as well and in fact we have

$$\nabla_H \mathbb{E}_n f = \mathbb{E}_n(P_n \nabla_H f) \qquad \forall n \in \mathbb{N},$$

where the right-hand side has to be understood as a Bochner H-valued integral. Indeed, by (10.1.2) we have

$$\nabla_H \mathbb{E}_n f_k(x) = \int_X P_n \nabla_H f_k(P_n x + (I - P_n)y)\gamma(dy)$$

for every $k \in \mathbb{N}$. The left hand side converges to $\nabla_H \mathbb{E}_n f$ in $L^p(X,\gamma;H)$ as $k \to +\infty$. The right hand side converges to $\mathbb{E}_n P_n \nabla_H f$ as $k \to +\infty$ since

$$\begin{aligned}
&\int_X \left| \int_X P_n \nabla_H (f_k - f)(P_n x + (I - P_n)y)\gamma(dy) \right|_H^p \gamma(dx) \\
&\leq \int_X \int_X |\nabla_H (f_k - f)(P_n x + (I - P_n)y)|_H^p \, \gamma(dy)\gamma(dx) \\
&= \int_X |\nabla_H (f_k - f)(x)|_H^p \gamma(dx)
\end{aligned}$$

by Proposition (7.3.2). □

Regular L^p cylindrical functions with L^p gradient are in $W^{1,p}(X,\gamma)$, see Exercise 10.2. The simplest nontrivial examples of Sobolev functions are the elements of X^*_γ.

Lemma 10.1.3. *$X^*_\gamma \subset W^{1,p}(X,\gamma)$ for every $p \in [1,+\infty)$, and $\nabla_H \hat{h} = h$ (constant) for every $\hat{h} \in X^*_\gamma$.*

Proof. Fix $p \geq 1$. For every $\hat{h} \in X^*_\gamma$, there exists a sequence $\ell_n \in X^*$ such that $\lim_{n\to\infty} \ell_n = \hat{h}$ in $L^2(X,\gamma)$. For every n, $m \in \mathbb{N}$ we have

$$\|\ell_n - \ell_m\|^p_{L^p(X,\gamma)} = \int_{\mathbb{R}} |\xi|^p \mathscr{N}(0, \|\ell_n - \ell_m\|^2_{L^2(X,\gamma)})(d\xi) = \int_{\mathbb{R}} |\tau|^p \mathscr{N}(0,1)\,(d\tau) \|\ell_n - \ell_m\|^p_{L^2(X,\gamma)}$$

so that (ℓ_n) is a Cauchy sequence in $L^p(X,\gamma)$. Its L^2-limit $\hat{h}$ coincides with its L^p-limit, if $p \neq 2$.

As ℓ_n is in X^*, $\nabla_H \ell_n$ is constant and it coincides with $R_\gamma \ell_n$, see (9.3.1). Since $\lim_{n\to\infty} \ell_n = \hat{h}$ in $L^2(X,\gamma)$ and R_γ is an isometry from X^*_γ to H, $H - \lim_{n\to\infty} R_\gamma \ell_n = R_\gamma \hat{h} = h$. Therefore,

$$\int_X |\nabla_H \ell_n - h|_H^p d\gamma = |R_\gamma \ell_n - h|_H^p \to 0 \quad \text{as n} \to \infty.$$

It follows that $\hat{h} \in W^{1,p}(X,\gamma)$ and $\nabla_H \hat{h} = h$. □

An important example of Sobolev functions is given by Lipschitz functions. Since a Lipschitz function is continuous, then it is Borel measurable.

Proposition 10.1.4. *If $f : X \to \mathbb{R}$ is Lipschitz continuous, then $f \in W^{1,p}(X, \gamma)$ for any $1 \le p < +\infty$.*

Proof. Let $L > 0$ be such that

$$|f(x) - f(y)| \le L\|x - y\| \qquad \forall\, x, y \in X.$$

Since $|f(x)| \le |f(0)| + L\|x\|$, by Theorem 2.3.1 (Fernique) $f \in L^p(X, \gamma)$ for any $1 \le p < \infty$.

Let us consider the conditional expectation $\mathbb{E}_n f$.

Let us notice that

$$\mathbb{E}_n f(x) = v_n(\hat{h}_1(x), \ldots, \hat{h}_n(x)),$$

with $v_n : \mathbb{R}^n \to \mathbb{R}$ an L_1–Lipschitz function since

$$\begin{aligned}
|v_n(z+\eta) - v_n(z)| =& \left|\mathbb{E}_n f\Big(\sum_{i=1}^n z_i h_i + \sum_{i=1}^n \eta_i h_i\Big) - \mathbb{E}_n f\Big(\sum_{i=1}^n z_i h_i\Big)\right| \\
\le& \int_X \left| f\Big(\sum_{i=1}^n z_i h_i + \sum_{i=1}^n \eta_i h_i\Big) + (I - P_n)y\Big) - f\Big(\sum_{i=1}^n z_i h_i + (I-P_n)y\Big)\right| \gamma(dy) \\
\le& L_1 \left|\sum_{i=1}^n \eta_i h_i\right|_H = L_1 |\eta|_{\mathbb{R}^n},
\end{aligned}$$

where we have used (3.1.3), $\|h\| \le c|h|_H$ for $h \in H$, and we have set $L_1 := cL$. By the Rademacher Theorem, v_n is differentiable λ_n–a.e. in $\mathbb{R}^n$ and $|\nabla v_n(z)|_{\mathbb{R}^n} \le L_1$ for a.e. $z \in \mathbb{R}^n$. Hence $v_n \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} |\nabla v_n(z)|^p_{\mathbb{R}^n} \gamma_n(dz) \le L_1^p.$$

We use now the map $T_n : X \to \mathbb{R}^n$, $T_n(x) = (\hat{h}_1(x), \ldots, \hat{h}_n(x))$. If $x \in X$ is a point such that v_n is differentiable at $T_n(x)$, then

$$\partial_h \mathbb{E}_n f(x) = \begin{cases} 0 & \text{if } h \in F_n^\perp \\ \\ \nabla v_n(T_n(x)) \cdot T_n(h) & \text{if } h \in F_n. \end{cases}$$

As a consequence, we can write

$$|\nabla_H \mathbb{E}_n f(x)|_H^2 = \sum_{i=1}^\infty |\partial_i \mathbb{E}_n f(x)|^2 = \sum_{i=1}^n |\partial_i \mathbb{E}_n f(x)|^2 = |\nabla v_n(T_n(x))|^2_{\mathbb{R}^n}.$$

We claim that for γ–a.e. x v_n is differentiable at $T_n(x)$. Indeed, let $A \subset \mathbb{R}^n$ be such that $\lambda_n(A) = 0$ and v_n is differentiable at any point in $\mathbb{R}^n \setminus A$. Since $\gamma_n \ll \lambda_n$, $\gamma_n(A) = 0$ and then $\gamma(T_n^{-1}(A)) = 0$ because $\gamma \circ T_n^{-1} = \gamma_n$, see exercise 2.4. Hence v_n is differentiable at any point $T_n(x)$, where $x \in X \setminus T_n^{-1}(A)$.

We Know that $\mathbb{E}_n f \to f$ in $L^p(X,\gamma)$ and we have

$$\int_X |\nabla_H \mathbb{E}_n f(x)|_H^p \gamma(dx) = \int_X |\nabla v_n(T_n x)|_{\mathbb{R}^n}^p \gamma(dx) = \int_{\mathbb{R}^n} |\nabla v_n(z)|_{\mathbb{R}^n}^p \gamma_n(dz) \le L_1^p.$$

By Proposition 9.3.10(v) $f \in W^{1,p}(X,\gamma)$ for every $1 < p < \infty$ and by inclusion $f \in W^{1,1}(X,\gamma)$. □

Further properties of $W^{1,p}$ functions are presented in Exercises 10.3, 10.4, 10.5.

10.2 Sobolev spaces of H-valued functions

We recall the definition of Hilbert–Schmidt operators, see e.g. [DS2, §XI.6] for more information.

Definition 10.2.1. *Let H_1, H_2 be separable Hilbert spaces. A linear operator $A \in \mathcal{L}(H_1,H_2)$ is called a Hilbert–Schmidt operator if there exists an orthonormal basis $\{h_j : j \in \mathbb{N}\}$ of H_1 such that*

$$\sum_{j=1}^{\infty} \|Ah_j\|_{H_2}^2 < \infty. \tag{10.2.1}$$

If A is a Hilbert–Schmidt operator and $\{e_j : j \in \mathbb{N}\}$ is any orthonormal basis of H_1, $\{y_j : j \in \mathbb{N}\}$ is any orthonormal basis of H_2, then

$$\|Ae_j\|_{H_2}^2 = \sum_{k=1}^{\infty} \langle Ae_j, y_k\rangle_{H_2}^2 = \sum_{k=1}^{\infty} \langle e_j, A^* y_k\rangle_{H_2}^2$$

so that

$$\sum_{j=1}^{\infty} \|Ae_j\|_{H_2}^2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle e_j, A^* y_k\rangle_{H_2}^2 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \langle e_j, A^* y_k\rangle_{H_2}^2 = \sum_{k=1}^{\infty} \|A^* y_k\|_{H_1}^2.$$

So, the convergence of the series (10.2.1) and the value of its sum are independent of the basis of H_1. We denote by $\mathcal{H}(H_1,H_2)$ the space of the Hilbert–Schmidt operators from H_1 to H_2, and we set

$$\|A\|_{\mathcal{H}(H_1,H_2)} = \left(\sum_{j=1}^{\infty} \|Ah_j\|_{H_2}^2\right)^{1/2},$$

for every orthonormal basis $\{h_j : j \in \mathbb{N}\}$ of H_1. Notice that if $H_1 = \mathbb{R}^n$, $H_2 = \mathbb{R}^m$, the Hilbert–Schmidt norm of any linear operator coincides with the Euclidean norm of the associated matrix.

The norm (10.2.1) comes from the inner product

$$\langle A, B\rangle_{\mathcal{H}(H_1,H_2)} = \sum_{j=1}^{\infty} \langle Ah_j, Bh_j\rangle_{H_2},$$

where for every couple of Hilbert–Schmidt operators A, B, the series in the right-hand side converges for every orthonormal basis $\{h_j : j \in \mathbb{N}\}$ of H_1, and its value is independent of the basis. The space $\mathcal{H}(H_1, H_2)$ is a separable Hilbert space with the above inner product.

If $H_1 = H_2 = H$, where H is the Cameron–Martin space of (X, γ), we set $\mathcal{H} := \mathcal{H}(H, H)$.

It is useful to generalise the notion of Sobolev space to H-valued functions. To this aim, we define the cylindrical E-valued functions as follows, where E is any normed space.

Definition 10.2.2. *For $k \in \mathbb{N}$ we define $\mathcal{F}C_b^k(X, E)$ (respectively, $\mathcal{F}C_b^\infty(X, E)$) as the linear span of the functions $x \mapsto v(x)y$, with $v \in \mathcal{F}C_b^k(X)$ (respectively, $v \in \mathcal{F}C_b^\infty(X)$) and $y \in E$.*

Therefore, every element of $\mathcal{F}C_b^k(X, E)$ may be written as

$$v(x) = \sum_{k=1}^{n} v_k(x) y_k \tag{10.2.2}$$

for some $n \in \mathbb{N}$, and $v_k \in \mathcal{F}C_b^k(X)$, $y_k \in E$. Such functions are Fréchet differentiable at every $x \in X$, with $v'(x) \in \mathcal{L}(X, E)$ defined by $v'(x)(h) = \sum_{k=1}^{n} v_k'(x)(h) y_k$ for every $h \in X$.

Similarly to the scalar case, we introduce the notion of H-differentiable function.

Definition 10.2.3. *A function $v : X \to E$ is called H-differentiable at $\overline{x} \in X$ if there exists $L \in \mathcal{L}(H, E)$ such that*

$$\|v(\overline{x} + h) - v(\overline{x}) - L(h)\|_E = o(|h|_H) \quad \textit{as } h \to 0 \textit{ in } H.$$

In this case we set $L =: D_H v(\overline{x})$.

If $v \in \mathcal{F}C_b^1(X, E)$ is given by (10.2.2), then it is H-differentiable at every $\overline{x} \in X$, and

$$D_H v(\overline{x})(h) = \sum_{k=1}^{n} [\nabla_H v_k(\overline{x}), h]_H \, y_k.$$

In particular, if $E = H$ and $\{h_j : j \in \mathbb{N}\}$ is any orthonormal basis of H we have

$$|D_H v(\overline{x})(h_j)|_H^2 \le \left(\sum_{k=1}^{n} |[\nabla_H v_k(\overline{x}), h_j]_H| \, |y_k|_H \right)^2 \le \sum_{k=1}^{n} [\nabla_H v_k(\overline{x}), h_j]_H^2 \sum_{k=1}^{n} |y_k|_H^2$$

so that $D_H v(\overline{x})$ is a Hilbert–Schmidt operator, and we have

$$\begin{aligned} |D_H v(\overline{x})|_{\mathcal{H}}^2 &= \sum_{j=1}^{\infty} |[D_H v(\overline{x}), h_j]_H|^2 \le \sum_{k=1}^{n} \sum_{j=1}^{\infty} |[\nabla_H v_k(\overline{x}), h_j]_H|^2 \sum_{k=1}^{n} |y_k|_H^2 \\ &= \sum_{k=1}^{n} |\nabla_H v_k(\overline{x})|_H^2 \sum_{k=1}^{n} |y_k|_H^2. \end{aligned}$$

Moreover, $x \mapsto \nabla_H v_k(x)$ is continuous and bounded for every k, therefore $x \mapsto D_H v(x)$ is continuous and bounded from X to $\mathcal{H}$. In particular, it belongs to $L^p(X, \gamma; \mathcal{H})$ for every $p \geq 1$.

The procedure to define Sobolev spaces of H-valued functions is similar to the procedure for scalar functions. Namely, we show that the operator D_H, seen as an unbounded operator from $L^p(X, \gamma; H)$ to $L^p(X, \gamma; \mathcal{H})$ with domain $\mathcal{F}C_b^1(X, H)$, is closable.

Lemma 10.2.4. *For every $p \geq 1$, the operator $D_H : \mathcal{F}C_b^1(X, H) \to L^p(X, \gamma; \mathcal{H})$ is closable in $L^p(X, \gamma; H)$.*

Proof. Let (V_n) be a sequence in $\mathcal{F}C_b^1(X, H)$ such that $V_n \to 0$ in $L^p(X, \gamma; H)$, with $D_H V_n \to \Phi$ in $L^p(X, \gamma; \mathcal{H})$. We have to show that $\Phi(x) = 0$ a.e., namely that

$$[\Phi(x)h_j, h_i]_H = 0, \forall i, j \in \mathbb{N},$$

a.e. in X.

Let $V_n(x) = \sum_{k=1}^{N(n)} v_k(x) y_k$. For every $j \in \mathbb{N}$ let us consider the functions $x \mapsto f_n(x) : [V_n(x), h_j]_H = \sum_{k=1}^{N(n)} v_k(x)[y_k, h_j]_H$. Each of them belongs to $\mathcal{F}C_b^1(X)$, and $f_n \to 0$ in $L^p(X, \gamma)$, since $|f_n(x)| \leq |V_n(x)|_H$. Moreover, $\nabla_H f_n(x) = \sum_{k=1}^{N(n)} \nabla_H v_k(x)[y_k, h_j]_H$ converges in $L^p(X, \gamma; H)$ to the vector field $\phi(x) = \sum_{i=1}^{\infty} [\Phi(x)h_i, h_j]_H h_i$. Indeed,

$$\nabla_H f_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{N(n)} [\nabla_H v_k(x), h_i]_H h_i [y_k, h_j]_H = \sum_{i=1}^{\infty} [D_H V_n(x)(h_i), h_j]_H$$

so that

$$\begin{aligned}
\int_X |\nabla_H f_n(x) - \phi(x)|_H^p d\gamma &= \int_X \left(\sum_{i=1}^{\infty} [D_H V_n(x)(h_i) - \Phi(x)h_i, h_j]_H^2 \right)^{p/2} d\gamma \\
&\leq \int_X \left(\sum_{i,l=1}^{\infty} [D_H V_n(x)(h_i) - \Phi(x)h_i, h_l]_H^2 \right)^{p/2} d\gamma \\
&= \int_X \left(\sum_{i=1}^{\infty} |(D_H V_n(x) - \Phi(x))(h_i)|_H^2 \right)^{p/2} d\gamma \\
&= \int_X |D_H V_n - \Phi|_{\mathcal{H}}^p d\gamma
\end{aligned}$$

that vanishes as $n \to \infty$. Since ∇_H is a closed operator in $L^p(X, \gamma)$, ϕ vanishes a.e. so that $[\Phi(x)h_i, h_j]_H = 0$ a.e. for every $i, j \in \mathbb{N}$. □

Definition 10.2.5. *For every $p \geq 1$ we define $W^{1,p}(X, \gamma; H)$ as the domain of the closure of the operator $D_H : \mathcal{F}C_b^1(X, H) \to L^p(X, \gamma; \mathcal{H})$ (still denoted by D_H) in $L^p(X, \gamma; H)$.*

Therefore, $W^{1,p}(X, \gamma; H)$ is a Banach space with the graph norm

$$\begin{aligned}
\|V\|_{W^{1,p}(X,\gamma;H)} &= \left(\int_X |V(x)|_H^p d\gamma \right)^{1/p} + \left(\int_X |D_H V(x)|_{\mathcal{H}}^p d\gamma \right)^{1/p} \\
&= \left(\int_X \left(\sum_{j=1}^{\infty} [V(x), h_j]_H^2 \right)^{p/2} d\gamma \right)^{1/p} + \left(\int_X \left(\sum_{i,j=1}^{\infty} [D_H V(x)(h_i), h_j]_H^2 \right)^{p/2} d\gamma \right)^{1/p}.
\end{aligned}$$

Let $v \in \mathcal{F}C_b^1(X,H)$,

$$v(x) = \sum_{k=1}^{n} \varphi_k(x) y_k,$$

with $\varphi_k \in \mathcal{F}C_b^1(X)$ and $y_k \in H$. Then v may be written in the form

$$v(x) = \sum_{j=1}^{\infty} v_j(x) h_j,$$

where the series converges in $W^{1,p}(X,\gamma;H)$. Indeed, setting

$$v_j(x) = [v(x), h_j]_H = \sum_{k=1}^{n} \varphi_k(x)[y_k, h_j]_H, \quad j \in \mathbb{N}$$

the sequence $s_m(x) = \sum_{j=1}^{m} v_j(x) h_j$ converges to v in $W^{1,p}(X,\gamma;H)$, because for each $k = 1,\dots,n$, the sequence $\sum_{j=1}^{m} \varphi_k(x)[y_k, h_j]_H$ converges to $\varphi_k(x) y$ in $W^{1,p}(X,\gamma;H)$. Moreover,

$$D_H v(x)(h) = \sum_{j=1}^{\infty} [\nabla_H v_j(x), h]_H h_j$$

so that, as in finite dimension,

$$[D_H v(x)(h_i), h_j]_H = [\nabla_H v_j(x), h_i]_H = \partial_i v_j(x).$$

10.2.1 The divergence operator

Let us recall the definition of adjoint operators. If X_1, X_2 are Hilbert spaces and $T : D(T) \subset X_1 \to X_2$ is a linear densely defined operator, an element $v \in X_2$ belongs to $D(T^*)$ iff the function $D(T) \to \mathbb{R}$, $f \mapsto \langle Tf, v\rangle_{X_2}$ has a linear continuous extension to the whole X_1, namely there exists $g \in X_1$ such that

$$\langle Tf, v\rangle_{X_2} = \langle f, g\rangle_{X_1}, \quad f \in D(T).$$

In this case g is unique (because $D(T)$ is dense in X_1) and we set

$$g = T^* v.$$

We are interested now in the case $X_1 = L^2(X,\gamma)$, $X_2 = L^2(X,\gamma;H)$ and $T = \nabla_H$. For $f \in W^{1,2}(X,\gamma)$, $v \in L^2(X,\gamma;H)$ we have

$$\langle Tf, v\rangle_{L^2(X,\gamma;H)} = \int_X [\nabla_H f(x), v(x)]_H \, \gamma(dx)$$

so that $v \in D(T^*)$ if and only if there exists $g \in L^2(X,\gamma)$ such that

$$\int_X [\nabla_H f(x), v(x)]_H \, \gamma(dx) = \int_X f(x) g(x) \, \gamma(dx), \quad f \in W^{1,2}(X,\gamma). \tag{10.2.3}$$

In this case, in analogy to the finite dimensional case, we set

$$g = -\mathrm{div}_\gamma v$$

and we call $-g$ *divergence* or *Gaussian divergence* of v. As $\mathcal{F}C_b^1(X)$ is dense in $W^{1,2}(X,\gamma)$, (10.2.3) is equivalent to

$$\int_X [\nabla_H f(x), v(x)]_H \,\gamma(dx) = \int_X f(x)g(x)\,\gamma(dx), \quad f \in \mathcal{F}C_b^1(X).$$

The main achievement of this section is the embedding $W^{1,2}(X,\gamma;H) \subset D(T^*)$. For its proof, we use the following lemma.

Lemma 10.2.6. *For every $f \in W^{1,2}(X,\gamma)$ and $h \in H$, $f\hat{h} \in L^2(X,\gamma)$ and*

$$\int_X (f\hat{h})^2 d\gamma \le 4\int_X (\partial_h f)^2 d\gamma + 2|h|_H^2 \int_X f^2\, d\gamma. \tag{10.2.4}$$

Proof. We already know that $\hat{h} \in W^{1,2}(X,\gamma)$. Then, for every $f \in \mathcal{F}C_b^1(X)$ we have $f^2\hat{h} \in W^{1,2}(X,\gamma)$ and

$$\begin{aligned}
\int_X (f\hat{h})^2 d\gamma &= \int_X (f^2\hat{h})\,\hat{h}\,d\gamma = \int_X \partial_h(f^2\hat{h})\,d\gamma \\
&= \int_X (2f\,\partial_h f\,\hat{h} + f^2\partial_h(\hat{h}))d\gamma \\
&= 2\int_X f\,\hat{h}\,\partial_h f\,d\gamma + |h|_H^2\int_X f^2\,d\gamma \\
&\le 2\Big(\int_X (f\hat{h})^2 d\gamma\Big)^{1/2}\Big(\int_X (\partial_h f)^2 d\gamma\Big)^{1/2} + |h|_H^2 \int_X f^2\,d\gamma.
\end{aligned}$$

Using the inequality $ab \le a^2/4 + b^2$, we get

$$\int_X (f\hat{h})^2 d\gamma \le \frac{1}{2}\int_X (f\hat{h})^2 d\gamma + 2\int_X (\partial_h f)^2 d\gamma + |h|_H^2 \int_X f^2\,d\gamma$$

so that f satisfies (10.2.4). Since $\mathcal{F}C_b^1(X)$ is dense in $W^{1,2}(X,\gamma)$, (10.2.4) holds for every $f \in W^{1,2}(X,\gamma)$. □

Theorem 10.2.7. *The Sobolev space $W^{1,2}(X,\gamma;H)$ is continuously embedded in $D(\mathrm{div}_\gamma)$ and the estimate*

$$\|\mathrm{div}_\gamma v\|_{L^2(X,\gamma)} \le \|v\|_{W^{1,2}(X,\gamma;H)}$$

holds. Moreover, fixing an orthonormal basis $\{h_n : n \in \mathbb{N}\}$ of H contained in $R_\gamma(X^*)$, and setting $v_n(x) = [v(x), h_n]_H$ for every $v \in W^{1,2}(X,\gamma;H)$ and $n \in \mathbb{N}$, we have

$$\mathrm{div}_\gamma v(x) = \sum_{n=1}^{\infty} \big(\partial_n v_n(x) - v_n(x)\hat{h}_n(x)\big),$$

where the series converges in $L^2(X,\gamma)$.

Proof. Consider a function $v \in W^{1,2}(X,\gamma;H)$ of the type

$$v(x) = \sum_{i=1}^{n} v_i(x)h_i, \quad x \in X. \tag{10.2.5}$$

with $v_i \in W^{1,2}(X,\gamma)$.

For every $f \in W^{1,2}(X,\gamma)$ we have $[\nabla_H f(x), v(x)]_H = \sum_{i=1}^{n} \partial_i f(x) v_i(x)$, so that

$$\int_X [\nabla_H f, v]_H d\gamma = \int_X \left(\sum_{i=1}^{n} \partial_i f\, v_i\right) d\gamma = \int_X \sum_{i=1}^{n} (-\partial_i v_i + v_i \hat{h}_i) f\, d\gamma$$

which yields

$$\mathrm{div}_\gamma v = \sum_{i=1}^{n} (\partial_i v_i - \hat{h}_i v_i).$$

Now we prove that

$$\int_X (\mathrm{div}_\gamma v)^2 d\gamma = \int_X |v|_H^2 d\gamma + \int_X \sum_{i,j=1}^{n} \partial_i v_j\, \partial_j v_i\, d\gamma, \tag{10.2.6}$$

showing, more generally, that if $u(x) = \sum_{i=1}^{n} u_i(x)h_i$ is another function of this type, then

$$\int_X (\mathrm{div}_\gamma v\, \mathrm{div}_\gamma u) d\gamma = \int_X [u,v]_H d\gamma + \int_X \sum_{i,j=1}^{n} \partial_i u_j\, \partial_j v_i\, d\gamma. \tag{10.2.7}$$

By linearity, it is sufficient to prove that (10.2.7) holds if the sums in u and v consist of a single addendum, $u(x) = f(x)h_i$, $v(x) = g(x)h_j$ for some f, $g \in W^{1,2}(X,\gamma)$ and i, $j \in \mathbb{N}$. In this case, (10.2.7) reads

$$\int_X (\partial_i f - \hat{h}_i f)(\partial_j g - \hat{h}_j g) d\gamma = \int_X fg\delta_{ij}\, d\gamma + \int_X \partial_j f\, \partial_i g\, d\gamma. \tag{10.2.8}$$

First, let f, $g \in \mathcal{F}C_b^2(X)$. Then,

$$\begin{aligned}
&\int_X (\partial_i f - \hat{h}_i f)(\partial_j g - \hat{h}_j g) d\gamma = -\int_X f\partial_i(\partial_j g - \hat{h}_j g) d\gamma \\
&= -\int_X f\partial_{ij} g\, d\gamma + \int_X fg\delta_{ij}\, d\gamma + \int_X f\hat{h}_j \partial_i g\, d\gamma \\
&= \int_X (\partial_j f - \hat{h}_j f)\partial_i g\, d\gamma + \int_X fg\delta_{ij}\, d\gamma + \int_X f\hat{h}_j \partial_i g\, d\gamma
\end{aligned}$$

so that (10.2.8) holds. Since $\mathcal{F}C_b^2(X)$ is dense in $W^{1,2}(X,\gamma)$, see Exercise 10.6, (10.2.8) holds for f, $g \in W^{1,2}(X,\gamma)$. Summing up, (10.2.7) follows, and taking $u = v$, (10.2.6) follows as well. Since the linear span of functions in (10.2.5) is dense in $W^{1,2}(X,\gamma;H)$ both equalities hold in the whole $W^{1,2}(X,\gamma;H)$. Notice also that (10.2.6) implies

$$\int_X (\mathrm{div}_\gamma v)^2 d\gamma \le \int_X |v|_H^2 d\gamma + \int_X \|D_H v\|_{\mathcal{H}}^2\, d\gamma. \tag{10.2.9}$$

If $v \in W^{1,2}(X,\gamma;H)$ we approximate it by the sequence

$$v_n(x) = \sum_{i=1}^{n} [v(x), h_i]_H h_i.$$

For every $f \in W^{1,2}(X,\gamma)$ we have

$$\int_X [\nabla_H f, v_n]_H d\gamma = -\int_X f \mathrm{div}_\gamma v_n \, d\gamma. \tag{10.2.10}$$

By estimate (10.2.9), $(\mathrm{div}_\gamma v_n)$ is a Cauchy sequence in $L^2(X,\gamma)$, so that it converges in $L^2(X,\gamma)$ to $g(x) := \sum_{j=1}^{\infty}(\partial_j v_j(x) - v_j(x)\hat{h}_j(x))$. Letting $n \to \infty$ in (10.2.10), we get

$$\int_X [\nabla_H f, v]_H d\gamma = -\int_X f\, g \, d\gamma,$$

so that $v \in D(T^*)$ and $\mathrm{div}_\gamma v = g$. □

Note that the domain of the divergence is larger than $W^{1,2}(X,\gamma;H)$, even in finite dimension. For instance, if $X = \mathbb{R}^2$ is endowed with the standard Gaussian measure, any vector field $v(x,y) = (\alpha_1(x) + \beta_1(y), \alpha_2(x) + \beta_2(y))$ with α_1, $\beta_2 \in W^{1,2}(\mathbb{R},\gamma_1)$, β_1, $\alpha_2 \in L^2(\mathbb{R},\gamma_1)$ belongs to the domain of the divergence, but it does not belong to $W^{1,2}(\mathbb{R}^2,\gamma_2;\mathbb{R}^2)$ unless also β_1, $\alpha_2 \in W^{1,2}(\mathbb{R},\gamma_1)$.

The divergence may be defined, still as a dual operator, also in a L^q context with $q \neq 2$. We recall that if X_1, X_2 are Banach spaces and : $D(T) \subset X_1 \to X_2$ is a linear densely defined operator, an element $v \in X_2^*$ belongs to $D(T^*)$ iff the function $D(T) \to \mathbb{R}$, $f \mapsto v(Tf)$ has a linear continuous extension to the whole X_1. Such extension is an element of X_1^*; denoting it by ℓ we have $\ell(f) = v(Tf)$ for every $f \in D(T)$.

We are interested in the case $X_1 = L^q(X,\gamma)$, $X_2 = L^q(X,\gamma;H)$, with $1 < q < \infty$, and $T : D(T) = W^{1,q}(X,\gamma)$, $Tf = \nabla_H f$. The dual space X_2^* consists of all the functions of the type

$$w \mapsto \int_X [w,v]_H d\gamma,$$

$v \in L^{q'}(X,\gamma;H)$, $q' = q/(q-1)$, see [DU, §IV.1], so we canonically identify $L^{q'}(X,\gamma;H)$ as $L^q(X,\gamma;H)^*$. We also identify $(L^q(X,\gamma))^*$ with $L^{q'}(X,\gamma)$. After these identifications, a function $v \in L^{q'}(X,\gamma;H)$ belongs to $D(T^*)$ iff there exists $g \in L^{q'}(X,\gamma)$ such that

$$\int_X [\nabla_H f(x), v(x)]_H \gamma(dx) = \int_X f(x)g(x)\gamma(dx), \quad \forall\, f \in W^{1,q}(X,\gamma),$$

which is equivalent to

$$\int_X [\nabla_H f(x), v(x)]_H \gamma(dx) = \int_X f(x)g(x)\gamma(dx), \quad \forall\, f \in \mathcal{F}C_b^1(X),$$

since $\mathcal{F}C_b^1(X)$ is dense in $W^{1,q}(X,\gamma)$. So, this is similar to the case $q = 2$, see (10.2.3).

Theorem 10.2.8. *Let $1 < q < \infty$, and let $T : D(T) = W^{1,q}(X,\gamma) \to L^q(X,\gamma;H)$, $Tf = \nabla_H f$. Then $W^{1,q}(X,\gamma;H) \subset D(T^*)$, and for every orthonormal basis $\{h_n:\ n\in\mathbb{N}\}$ of H we have*

$$T^*v(x) = -\sum_{n=1}^{\infty}(\partial_n v_n(x) - v_n(x)\hat{h}_n(x)), \quad v \in W^{1,q}(X,\gamma;H)$$

where $v_n(x) = [v(x), h_n]_H$, and the series converges in $L^q(X,\gamma)$.

The proof of Theorem 10.2.8 for $q \neq 2$ is not as easy as in the case $q = 2$. See [B, Prop. 5.8.8]. The difficult part is the estimate

$$\|T^*v\|_{L^q(X,\gamma)} \leq C\|v\|_{W^{1,q}(X,\gamma;H)},$$

even for good vector fields $v = \sum_{i=1}^n v_i(x)h_i$, with $v_i \in \mathcal{F}C_b^1(X)$.

We may still call "Gaussian divergence" the operator T^*.

10.3 The Sobolev spaces $W^{2,p}(X,\gamma)$

Let us start with regular functions, recalling the definition of the second order derivative $f''(x)$ given in Lecture 9. If $f : X \to \mathbb{R}$ is differentiable at any $x \in X$, we consider the function $X \to X^*$, $x \mapsto f'(x)$. If this function is differentiable at $\overline{x}$, we say that f is twice (Fréchet) differentiable at $\overline{x}$. In this case there exists $L \in \mathcal{L}(X, X^*)$ such that

$$\|f'(\overline{x}+h) - f'(\overline{x}) - Lh\|_{X^*} = o(\|h\|) \quad \text{as } h \to 0 \text{ in } X,$$

and we set $L =: f''(\overline{x})$.

In our setting we are interested in increments $h \in H$, and in H-differentiable functions. If $f : X \to \mathbb{R}$ is H-differentiable at any $x \in X$, we say that f is twice H-differentiable at $\overline{x}$ if there exists a linear operator $L_H \in \mathcal{L}(H)$ such that

$$|\nabla_H f(\overline{x}+h) - \nabla_H f(\overline{x}) - L_H h|_H = o(|h|_H) \quad \text{as } h \to 0 \text{ in } H.$$

The operator L_H is denoted by $D^2_H f(\overline{x})$.

We recall that if f is differentiable at x, it is also H-differentiable and we have $\nabla_H f(x) = R_\gamma f'(x)$. So, if f is twice differentiable at $\overline{x}$, with $f''(\overline{x}) = L$, then $D^2_H f(\overline{x})h = R_\gamma L h$. Indeed,

$$|R_\gamma f'(\overline{x}+h) - R_\gamma f'(\overline{x}) - R_\gamma Lh|_H \leq \|R_\gamma\|_{\mathcal{L}(X^*,H)}\|f'(\overline{x}+h) - f'(\overline{x}) - Lh\|_{X^*} = o(\|h\|)$$

as $h \to 0$ in X, and therefore,

$$|R_\gamma f'(\overline{x}+h) - R_\gamma f'(\overline{x}) - R_\gamma Lh|_H = o(|h|_H) \quad \text{as } h \to 0 \text{ in } H.$$

If $f \in \mathcal{F}C_b^2(X)$, $f(x) = \varphi(\ell_1(x), \ldots, \ell_n(x))$ with $\varphi \in C_b^2(\mathbb{R}^n)$, $\ell_k \in X^*$, then f is twice differentiable at any $\overline{x} \in X$ and

$$(f''(\overline{x})v)(w) = \sum_{i,j=1}^n \frac{\partial^2\varphi}{\partial x_i \partial x_j}(\ell_1(\overline{x}), \ldots, \ell_n(\overline{x})\ell_i(v)\ell_j(w), \quad v,\ w \in X$$

so that

$$[D^2_H f(\overline{x})h, k]_H = \sum_{i,j=1}^n \frac{\partial^2\varphi}{\partial x_i \partial x_j}(\ell_1(\overline{x}), \ldots, \ell_n(\overline{x})[R_\gamma \ell_i, h]_H \, [R_\gamma \ell_j, k]_H, \quad h,\ k \in H.$$

$D^2_H f(\overline{x})$ is a Hilbert–Schmidt operator, since for any orthonormal basis $\{h_j :\ j \in \mathbb{N}\}$ of H we have

$$\begin{aligned}
\sum_{m,k=1}^\infty [D^2_H f(\overline{x})h_m, h_k]^2_H &\leq \sum_{m,k=1}^\infty \bigg(\sum_{i,j=1}^n \frac{\partial^2\varphi}{\partial x_i \partial x_j}^2\bigg)\bigg(\sum_{i=1}^n [R_\gamma \ell_i, h_m]^2_H\bigg)\bigg(\sum_{j=1}^n [R_\gamma \ell_j, h_k]^2_H\bigg) \\
&= \|D^2_H \varphi\|_{\mathcal{H}(\mathbb{R}^n,\mathbb{R}^n)} \sum_{m,k=1}^\infty \sum_{i=1}^n [R_\gamma \ell_i, h_m]^2_H \sum_{j=1}^n [R_\gamma \ell_j, h_k]^2_H \\
&= \|D^2 \varphi\|_{\mathcal{H}(\mathbb{R}^n,\mathbb{R}^n)} \sum_{i=1}^n \sum_{m=1}^\infty [R_\gamma \ell_i, h_m]^2_H \sum_{j=1}^n \sum_{k=1}^\infty [R_\gamma \ell_j, h_k]^2_H \\
&= \|D^2 \varphi\|_{\mathcal{H}(\mathbb{R}^n,\mathbb{R}^n)} \sum_{i=1}^n |R_\gamma \ell_i|^2_H \sum_{j=1}^n |R_\gamma \ell_j|^2_H
\end{aligned}$$

where the derivatives of φ are evaluated at $(\ell_1(\overline{x}), \ldots, \ell_n(\overline{x}))$. Since $\|D^2\varphi\|_{\mathcal{H}(\mathbb{R}^n,\mathbb{R}^n)}$ is bounded, $x \to \|D^2_H f(x)\|_{\mathcal{H}}$ is bounded in X.

The next lemma is similar to Lemma 9.3.7. We omit the proof.

Lemma 10.3.1. *For every $p \geq 1$, the operator*

$$(\nabla_H, D^2_H) : \mathcal{F}C^2_b(X) \to L^p(X, \gamma; H) \times L^p(X, \gamma; \mathcal{H})$$

is closable in $L^p(X, \gamma)$.

Definition 10.3.2. *For every $p \geq 1$, $W^{2,p}(X, \gamma)$ is the domain of the closure of*

$$(\nabla_H, D^2_H) : \mathcal{F}C^2_b(X) \to L^p(X, \gamma; H) \times L^p(X, \gamma; \mathcal{H})$$

in $L^p(X, \gamma)$. Therefore, $f \in L^p(X, \gamma)$ belongs to $W^{2,p}(X, \gamma)$ iff there exists a sequence $(f_n) \subset \mathcal{F}C^2_b(X)$ such that $f_n \to f$ in $L^p(X, \gamma)$, $\nabla_H f_n$ converges in $L^p(X, \gamma; H)$ and $D^2_H f_n$ converges in $L^p(X, \gamma; \mathcal{H})$. In this case we set $D^2_H f := \lim_{n\to\infty} D^2_H f_n$.

$W^{2,p}(X, \gamma)$ is a Banach space with the graph norm

$$\begin{aligned}
\|f\|_{W^{2,p}} &:= \|f\|_{L^p(X,\gamma)} + \|\nabla_H f\|_{L^p(X,\gamma;H)} + \|D^2_H f\|_{L^p(X,\gamma;\mathcal{H})} \qquad (10.3.1) \\
&= \bigg(\int_X |f|^p d\gamma\bigg)^{1/p} + \bigg(\int_X |\nabla_H f|^p_H d\gamma\bigg)^{1/p} + \bigg(\int_X |D^2_H f|^p_{\mathcal{H}} d\gamma\bigg)^{1/p}.
\end{aligned}$$

Fixed any orthonormal basis $\{h_j :\ j \in \mathbb{N}\}$ of H, for every $f \in W^{2,p}(X, \gamma)$ we set

$$\partial_{ij} f(x) = [D^2_H f(x)h_j, h_i]_H.$$

For every sequence of approximating functions f_n we have

$$[D_H^2 f_n(x)h_j, h_i]_H = [D_H^2 f_n(x)h_i, h_j]_H, \quad x \in X,\ i, j \in \mathbb{N},$$

then the equality

$$\partial_{ij} f(x) = \partial_{ji} f(x), \quad \text{a.e.}$$

holds. Therefore, the $W^{2,p}$ norm may be rewritten as

$$\left(\int_X |f|^p d\gamma\right)^{1/p} + \left(\int_X \left(\sum_{j=1}^\infty (\partial_j f)^2\right)^{p/2} d\gamma\right)^{1/p} + \left(\int_X \left(\sum_{i,j=1}^\infty (\partial_{ij} f)^2\right)^{p/2} d\gamma\right)^{1/p}.$$

Let X be a Hilbert space and assume that γ is nondegenerate. Then, another class of $W^{2,p}$ spaces looks more natural. As in Remark 9.3.11, we may replace $(\nabla_H f, D_H^2 f)$ in Definition 10.3.2 by $(\nabla f, f'')$. The proof of Lemma 10.3.1 works as well with this choice. So, we define $\widetilde{W}^{2,p}(X,\gamma)$ as the domain of the closure of the operator $T : \mathcal{F}C_b^2(X) \to L^p(X,\gamma;X) \times L^p(X,\gamma;\mathcal{H}(X,X))$, $f \mapsto (\nabla f, f'')$ in $L^p(X,\gamma)$ (still denoted by T), and we endow it with the graph norm of T. This space is much smaller than $W^{2,p}(X,\gamma)$ if X is infinite dimensional. Indeed, fix as usual any orthonormal basis $\{e_j : \ j \in \mathbb{N}\}$ of X consisting of eigenvectors of Q, $Qe_j = \lambda_j e_j$, and set $h_j = \sqrt{\lambda_j} e_j$. Then $\{h_j : \ j \in \mathbb{N}\}$ is a orthonormal basis of H, $\partial_j f(x) = \sqrt{\lambda_j}\partial f/\partial e_j$, $\partial_{ij} f(x) = \sqrt{\lambda_i \lambda_j}\partial^2 f/\partial e_i \partial e_j$, and

$$\begin{aligned}\|f\|_{W^{2,p}(X,\gamma)} =& \|f\|_{L^p(X,\gamma)} + \left(\int_X \left(\sum_{j=1}^\infty \lambda_j \left(\frac{\partial f}{\partial e_j}\right)^2\right)^{p/2} d\gamma\right)^{1/p} \\ &+ \left(\int_X \left(\sum_{i,j=1}^\infty \lambda_i\lambda_j \left(\frac{\partial^2 f}{\partial e_i \partial e_j}\right)^2\right)^{p/2} d\gamma\right)^{1/p},\end{aligned}$$

while

$$\begin{aligned}\|f\|_{\widetilde{W}^{2,p}(X,\gamma)} =& \|f\|_{L^p(X,\gamma)} + \left(\int_X \left(\sum_{j=1}^\infty \left(\frac{\partial f}{\partial e_k}\right)^2\right)^{p/2} d\gamma\right)^{1/p} \\ &+ \left(\int_X \left(\sum_{i,j=1}^\infty \left(\frac{\partial^2 f}{\partial e_i \partial e_j}\right)^2\right)^{p/2} d\gamma\right)^{1/p}.\end{aligned}$$

Since $\lim_{j\to\infty} \lambda_j = 0$, the $\widetilde{W}^{2,p}(X,\gamma)$ norm is stronger than the $W^{2,p}(X,\gamma)$ norm. In particular, the function $f(x) = \|x\|^2$ belongs to $W^{2,p}(X,\gamma)$ for every $p \geq 1$ but it does not belong to $\widetilde{W}^{2,p}(X,\gamma)$ for any $p \geq 1$, because $f''(x) = 2I$ for every $x \in X$ and $\partial^2 f/\partial e_i \partial e_j = 2\delta_{ij}$.

10.4 Exercises

Exercise 10.1. Prove that (10.1.2) holds.

Exercise 10.2. Prove that if $f \in \mathcal{F}C^1(X) \cap L^p(X,\gamma)$, $1 \le p < \infty$ and $\nabla_H f \in L^p(X,\gamma)$ then $f \in W^{1,p}(X,\gamma)$.

Exercise 10.3. Prove that if $f \in W^{1,p}(X,\gamma)$ then $f^+, f^-, |f| \in W^{1,p}(X,\gamma)$ as well. Compute $\nabla_H f^+, \nabla_H f^-, \nabla_H |f|$ and deduce that $\nabla_H f = 0$ a.e. on $\{f = c\}$ for every $c \in \mathbb{R}$.

Exercise 10.4. Let $\varphi \in W^{1,p}(\mathbb{R}^n, \gamma_n)$ and let $\ell_1, \dots, \ell_n \in X^*$, with $\langle \ell_i, \ell_j \rangle_{L^2(X,\gamma)} = \delta_{ij}$. Prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = \varphi(\hat{h}_1(x), \dots, \hat{h}_n(x))$ belongs to $W^{1,p}(X,\gamma)$.

Exercise 10.5. Let $f \in L^p(X,\gamma)$, $p > 1$, be such that $\mathbb{E}_n f \in W^{1,p}(X,\gamma)$ for every $n \in \mathbb{N}$, with $\sup_n \|\nabla_H \mathbb{E}_n f\|_{L^p(X,\gamma;H)} < \infty$. Prove that $f \in W^{1,p}(X,\gamma)$.

Exercise 10.6. Prove that $\mathcal{F}C^2_b(X)$ is dense in $W^{1,2}(X,\gamma)$

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