Exercise 1.1. Let μ be a finite positive measure on $(X, \mathcal{F})$. Then, it satisfies the monotonicity property, and recall that if $A \subset B$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$. Indeed $B = A \cup (B \setminus A)$, and from the properties of μ it follows that $\mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A)$.

- If $\{E_n\}_{n\in\mathbb{N}}$ is an increasing sequence of subsets of X, that is $E_n \subset E_{n+1}$, for any $n \in \mathbb{N}$, we have

$$E_n = \bigcup_{h=1}^{n} F_h, \qquad \bigcup_{n=1}^{+\infty} E_n = \bigcup_{h=1}^{+\infty} F_h,$$

for any $n \in \mathbb{N}$, where $F_1 = E_1$ and $F_h := E_h \setminus E_{h-1}$, for any $h \geq 2$. Hence from the properties of μ

$$\mu(E_n) = \sum_{h=1}^{n} \mu(F_h),$$

and since $\mu(E_n)$ is increasing, the limit of $\mu(E_n)$ as $n \to +\infty$ exists. Passing to the limit we get

$$\lim_{n\to+\infty} \mu(E_n) = \sum_{h=1}^{+\infty} \mu(F_h) = \mu\left(\bigcup_{h=1}^{+\infty} F_h\right) = \mu\left(\bigcup_{h=1}^{+\infty} E_h\right),$$

where the second equality follows from the σ-additivity of μ.

- Let $\{E_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of subsets of X, that is $E_{n+1} \subset E_n$, for any $n \in \mathbb{N}$. From the De Morgan laws we have

$$\bigcap_{n=1}^{+\infty} E_n = \left(\bigcup_{n=1}^{+\infty} E_n^c\right)^c = X \setminus \bigcup_{n=1}^{+\infty} E_n^c.$$

Since the sequence $\{E_n^c\}_{n\in\mathbb{N}}$ is increasing, from part (i) it follows that

$$\begin{aligned}
\mu\left(\bigcap_{n=1}^{+\infty} E_n\right) &= \mu(X) - \mu\left(\bigcup_{n=1}^{+\infty} E_n^c\right) \\
&= \mu(X) - \lim_{n\to+\infty} \mu(E_n^c) \\
&= \mu(X) - \lim_{n\to+\infty} (\mu(X) - \mu(E_n)) \\
&= \lim_{n\to+\infty} \mu(E_n),
\end{aligned}$$

and the last term makes sense because the sequence $\{\mu(E_n)\}_{n\in\mathbb{N}}$ is decreasing in $\mathbb{R}$.

Exercise 1.2.

- Let μ be a finite real measure. For all $E \in \mathcal{F}$ we define

$$|\mu|(E) := \sup\left\{\sum_{k=1}^{\infty} |\mu(E_k)| : \ E_k \in \mathcal{F} \text{ pairwise disjoint, } E = \bigcup_{k=1}^{\infty} E_k\right\} \tag{1}$$

We prove that $|\mu|$ is positive finite measure.

First we prove that $|\mu|$ is a finite measure.

Since μ is a finite real measure, there exist μ_1, μ_2 positive finite measures such that $\mu = \mu_1 - \mu_2$. Let $E \in \mathcal{F}$, then for all $\varepsilon > 0$ there exists a partition $\{E_h\}_{h\geq 1} \subset \mathcal{F}$ of E such that

$$\begin{aligned}|\mu|(E) &\leq \sum_{h=1}^{\infty} |\mu(E_h)| + \varepsilon = \sum_{h=1}^{\infty} |\mu_1(E_h) - \mu_2(E_h)| + \varepsilon \\ &\leq \sum_{h=1}^{\infty} \mu_1(E_h) + \sum_{h=1}^{\infty} \mu_2(E_h) + \varepsilon = \mu_1(E) + \mu_2(E) + \varepsilon\end{aligned}$$

then the total variation $|\mu|$ is finite.

Now we prove that $|\mu|$ is a positive measure and $|\mu|(\emptyset) = 0$.

The positivity of $|\mu|$ follows by definition (1). If $\{E_n\}_{n\geq 1}$ is a measurable partition of $\emptyset$, then $E_n = \emptyset$ for all $n \geq 1$, therefore $|\mu|(\emptyset) = 0$.

Finally we prove that $|\mu|$ is σ-additive.

Let $E \in \mathcal{F}$. We define $E_1 = E$ and $E_n = \emptyset$ for $n \geq 2$. Therefore $\{E_n\}_{n\geq 1}$ is a measurable partition of E. By definition (1), we have

$$\sum_{n=1}^{\infty} |\mu(E_n)| \leq |\mu|(E)$$

then we get $|\mu(E)| \leq |\mu|(E)$.

Let $\{E_n\}_{n\geq 1}$ a measurable partition of E. Let $\varepsilon > 0$, and for each j, choose a partition $\{A_{i,n}\}_{i\geq 1}$ of E_n such that

$$|\mu|(E_n) \leq \sum_{i=1}^{\infty} |\nu(A_{i,n})| + \frac{\varepsilon}{2^n}.$$

Then

$$\sum_{n=1}^{\infty} |\mu|(E_n) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} |\nu(A_{i,n})| + \varepsilon \leq |\nu|(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\sum_{n=1}^{\infty} |\mu|(E_n) \leq |\nu|(E) = |\nu| \left(\bigcup_{n=1}^{\infty} E_n \right).$$

For the reverse inequality, let $\{A_i\}_{i\geq 1}$ be another partition of E. Since $\{A_i \cap E_n\}_{i\geq 1}$ is a measurable partition of E_n and $\{A_i \cap E_n\}_{n\geq 1}$ is a measurable partition of A_i, we have

$$\begin{aligned}\sum_{i=1}^{\infty} |\mu(A_i)| &= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \mu(A_i \cap E_n) \right| \leq \sum_{i,n=1}^{\infty} |\mu(A_i \cap E_n)| \\ &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} |\mu|(A_i \cap E_n) \leq \sum_{i=1}^{\infty} |\mu|(E_n).\end{aligned}$$

Taking the supremum over all measurable partitions $\{A_i\}_{i\geq 1}$ of E we get

$$|\mu|(E) \leq \sum_{i=1}^{\infty} |\mu|(E_n).$$

- Let μ be a positive measure and $f \in L^1(X,\mu)$, and let $\nu := f\mu$ be the measure defined by

$$\nu(B) = \int_B f d\mu, \forall B \in \mathcal{F}.$$

We prove that

$$|\nu|(B) = \int_B |f| d\mu, \forall B \in \mathcal{F}.$$

We recall that ν is absolutely continuous with respect to $|\nu|$; then there exists a unique real valued function $h \in L^1(X, |\nu|)$ such that $\nu = h|\nu|$ and $|h| = 1$ $|\nu|$-a.e., therefore for all $E \in \mathcal{F}$ we get

$$\nu(E) = \int_E h \; d|\nu|.$$

For any $E, F \in \mathcal{F}$ we have

$$\int_E f\chi_F \; d\mu = \int_{E\cap F} f \; d\mu = \nu(E \cap F) = \int_E h\chi_F \; d|\nu|.$$

Given $g : X \to \mathbb{R}$ bounded and measurable, we claim that:

$$\forall E \in \mathcal{F}, \; \int_E fg \; d\mu = \int_E hg \; d|\nu| \tag{2}$$

We know that equation (2) holds for $g = \chi_F$ with $F \in \mathcal{F}$. By the linearity of the integral, (2) is also true whenever g is a simple function on $(X, \mathcal{F})$. Let g be nonnegative, measurable and bounded, then there exists a sequence $\{s_n\}_{n\geq 1}$ of simple functions on $(X, \mathcal{F})$ such that $s_n \uparrow g$. Having proved (2) for simple functions, for all $n \geq 1$ we have

$$\int_X \chi_E f s_n \; d\mu = \int_X \chi_E h s_n \; d|\nu| \tag{3}$$

From $s_n \to g$ we get: $\chi_E f s_n \to \chi_E f g$ and $\chi_E h s_n \to \chi_E hg$. Since $|\chi_E f s_n| \leq |f|g \in L^1(X,\mu)$ (since g is bounded) and $|\chi_E h s_n| \leq |h|g \in L^1(X, |\nu|)$, it follows from the dominated convergence theorem that

$$\int_X \chi_E f s_n \; d\mu \to \int_X \chi_E fg \; d\mu \quad \text{and} \quad \int_X \chi_E h s_n \; d|\nu| \to \int_X \chi_E hg \; d|\nu|$$

as $n \to \infty$. Taking the limit in (3) as $n \to \infty$, we see that (2) holds for all non-negative measurable, bounded g. If g is now an arbitrary $\mathbb{C}$-valued map which is measurable and bounded, then it can be expressed as $g = g_1 - g_2 + i(g_3 - g_4)$ where each g_i is non-negative, measurable and bounded. From the linearity of the integral, we conclude that (2) is also true for g, which completes the proof of our initial claim.

Since $|h| = 1$, applying (2) to $g = \text{sign}(h)$, for all $E \in \mathcal{F}$ we get

$$\int_E f \,\text{sign}(h)\; d\mu = \int_E h \,\text{sign}(h)\; d|\nu| = \int_E d|\nu| = |\nu|(E).$$

The total variation $|\nu|$, of the finite real measure ν, has values in $\mathbb{R}^+$, hence:

$$\int_E f \,\text{sign}(h) d\mu \geq 0$$

Define $\varphi = f \,\text{sign}(h)$. Then $\varphi \in L^1(X, \mu)$, moreover for all $E \in \mathcal{F}$ we have

$$\int_E \varphi \; d\mu \geq 0.$$

Taking $E = \{\varphi < -1/n\}$ for some $n \geq 1$ we get

$$0 \leq \int_E \varphi \; d\mu \leq -\frac{1}{n}\mu(\{\varphi < -1/n\}) \leq 0$$

from which we see that $\mu(\{\varphi < -1/n\}) = 0$ for all $n \geq 1$. Since

$$\{\varphi < 0\} \subset \bigcup_{n \geq 1} \{\varphi < -1/n\}$$

it follows that $\mu(\{\varphi < 0\}) = 0$ and consequently, $\varphi \in \mathbb{R}^+$ μ-a.e., that is $f \,\text{sign}(h) \geq 0$ a.e..

Then there exists $N \in \mathcal{F}$ with $\mu(N) = 0$ and $f(x) \,\text{sign}(h)(x) \in \mathbb{R}^+$ for all $x \in X \setminus N$. In particular, since $|h(x)| = 1$, for all $x \in X \setminus N$ we have

$$f(x) \,\text{sign}(h)(x) = |f(x) \,\text{sign}(h)(x)| = |f(x)|.$$

It follows that $f \,\text{sign}(h) = |f|$ μ−a.e..

Therefore for all $E \in \mathcal{F}$ we have

$$|\nu|(E) = \int_E f \,\text{sign}(h)\; d\mu = \int_E |f|\; d\mu.$$

- Now we prove that if $\mu \perp \nu$ then $|\mu + \nu| = |\mu| + |\nu|$.

  Since $\mu \perp \nu$ then there exist $A \in \mathcal{F}$ such that $|\mu|(A) = 0$ and $|\nu|(X \setminus A) = 0$. For $E \in \mathcal{F}$ we have $E = (E \cap A) \cup (E \cap (X \setminus A))$ and $(E \cap A) \cap (E \cap (X \setminus A)) = \emptyset$. By the additivity we get

$$\begin{aligned} |\mu + \nu|(E) &= |\mu + \nu|(E \cap A) + |\mu + \nu|(E \cap (X \setminus A)) \\ &= \sup\left\{ \sum_{k=1}^{\infty} |\mu(B_k) + \nu(B_k)| : \; E \cap A = \bigcup_{k=1}^{\infty} B_k \right\} + \\ &\quad + \sup\left\{ \sum_{k=1}^{\infty} |\mu(F_k) + \nu(F_k)| : \; E \cap (X \setminus A) = \bigcup_{k=1}^{\infty} F_k \right\}. \end{aligned}$$

We recall that $|\mu(A)| \le |\mu|(A) = 0$ so that $\mu(A) = 0$; since $B_k \subset A$ for all $k \ge 1$ we have $\mu(B_k) = 0$ for all $k \ge 1$. Similarly $\nu(F_k) = 0$ for all $k \ge 1$, then

$$\begin{aligned} |\mu+\nu|(E) = \sup\left\{\sum_{k=1}^{\infty} |\nu(B_k)| :\ E \cap A = \bigcup_{k=1}^{\infty} B_k\right\} + \\ + \sup\left\{\sum_{k=1}^{\infty} |\mu(F_k)| :\ E \cap (X \setminus A) = \bigcup_{k=1}^{\infty} F_k\right\} \\ = |\nu|(E \cap A) + |\mu|(E \cap (X \setminus A)). \end{aligned}$$

Now we recall that $|\nu|(E \cap (X \setminus A)) = 0 = |\mu|(E \cap A)$, then

$$|\mu+\nu|(E) = |\mu|(E) + |\nu|(E).$$

Exercise 1.4. Prove the Vitali–Lebesgue Theorem 1.1.7
Proof. We will need the following two well known results (see Rudin, *Real and Complex Analysis*):
Theorem 1(Riesz's Theorem). Let $(X, \mathscr{F})$ be a measurable space and let μ be a positive finite measure on it. Every sequence (f_k) of measurable functions which converges in measure to a function f, contains a subsequence converging to f pointwise μ-a.e.
Theorem 2(Egorov's Theorem). Let $(X, \mathscr{F})$ be a measurable space and let μ be a positive finite positive measure on it. If (f_k) is a sequence of measurable functions which converges pointwise μ-a.e. on X, then for every $\epsilon > 0$ a measurable set $E_\epsilon \subseteq X$ exists such that $\mu(E_\epsilon) < \epsilon$ and f_k converges uniformly on $X \setminus E_\epsilon$.

Let (f_k) a sequence satisfying the hypotheses of the Vitali–Lebesgue Theorem, by Theorem 1 we can assume that f_k converges pointwise μ-a.e. to f.

Let $\epsilon > 0$ and fix $M_0 > 0$ such that

$$\sup_{k\in\mathbb{N}} \int_{|f_k|>M} |f_k| d\mu < \epsilon \qquad \text{for every } M \ge M_0$$

By Fatou's Lemma we get

$$\begin{aligned} \int_X |f| d\mu \le \liminf_{k\to+\infty} \int_X |f_k| d\mu \le \sup_{k\in\mathbb{N}} \int_X |f_k| d\mu \le \\ \le \sup_{k\in\mathbb{N}} \int_{|f_k|\le M_0} |f_k| d\mu + \sup_{k\in\mathbb{N}} \int_{|f_k|>M_0} |f_k| d\mu \le M_0 \mu(X) + \epsilon. \end{aligned}$$

Thus $f \in L^1(X, \mu)$, in particular there exists $\overline{M} > M_0$ such that $\int_{|f|>\overline{M}} |f| d\mu < \varepsilon$. Now let E_ϵ the sets obtained applying Egorov's Theorem and $k \ge k_0$ such that $\sup_{x\in X\smallsetminus E_\varepsilon}|f_k-f|<\varepsilon$. Observe that

$$\begin{aligned}\int_X |f_k-f|d\mu &= \int_{X\smallsetminus E_\epsilon}|f_k-f|d\mu+\int_{E_\epsilon}|f_k-f|d\mu \le\\ &\le \mu(X\smallsetminus E_\epsilon)\sup_{x\in X\smallsetminus E_\varepsilon}|f_k-f|+\int_{E_\epsilon}|f_k|d\mu+\int_{E_\epsilon}|f|d\mu\le\\ &\le \mu(X)\epsilon+\int_{E_\epsilon\cap\{|f_k|\le M_0\}}|f_k|d\mu+\int_{E_\epsilon\cap\{|f_k|>M_0\}}|f_k|d\mu+\\ &+\int_{E_\epsilon\cap\{|f|\le\overline{M}\}}|f|d\mu+\int_{E_\epsilon\cap\{|f|>\overline{M}\}}|f|d\mu\le\\ &\le \mu(X)\epsilon+M_0\varepsilon+\varepsilon+\overline{M}\varepsilon+\varepsilon.\end{aligned}$$

Exercise 1.5. Let $\mathcal{F}_1\times\mathcal{F}_2$ be the σ-fields generated by $\mathcal{G}:=\{E_1\times E_2 : E_i\in X_i, i=1,2\}$, and, for any $x\in X_1$ and $y\in X_2$, let us set $E_x:=\{y\in X_2:(x,y)\in E\}$ and $E^y:=\{x\in X_1:(x,y)\in E\}$. We want to prove that the families of sets

$$\mathcal{G}_x:=\{F\in\mathcal{F}_1\times\mathcal{F}_2 : F_x\in\mathcal{F}_2\},\quad \mathcal{G}^y:=\{F\in\mathcal{F}_1\times\mathcal{F}_2 : F^y\in\mathcal{F}_1\},$$

are σ-fields and contain $\mathcal{G}$. Since the proofs for $\mathcal{G}_x$ and $\mathcal{G}^y$ are analogous, we consider only $\mathcal{G}_x$.

Let $x\in X$. We show that $\mathcal{G}_x$ is a σ-field.

(i) $\emptyset, X_1\times X_2\in\mathcal{G}_x$. Indeed, we have $\emptyset_x=\{y\in X_2:(x,y)\in\emptyset\}=\emptyset\in\mathcal{F}_2$, and $(X_1\times X_2)_x=\{y\in X_2:(x,y)\in X_1\times X_2\}=X_2\in\mathcal{F}_2$.

(ii) If $F\in\mathcal{G}_x$, then $F^c\in\mathcal{G}_x$. Since $F\in\mathcal{G}_x$, $F_x\in\mathcal{F}_2$ and so $(F_x)^c\in\mathcal{F}_2$. We note that $F^c\in\mathcal{G}_x$ is equivalent to the condition $(F^c)_x\in\mathcal{F}_2$. We have

$$\begin{aligned}(F^c)_x &:= \{y\in X_2:(x,y)\in F^c\}=\{y\in X_2:(x,y)\notin F\}\\ &= \{y\in X_2:(x,y)\in F\}^c=(F_x)^c\in\mathcal{F}_2.\end{aligned}$$

(iii) If $\{F_n\}_{n\in\mathbb{N}}\subset\mathcal{G}_x$, then $\bigcup_{n=1}^{+\infty}F_n\in\mathcal{G}_x$. This is true if and only if

$$\left\{y\in X_2:(x,y)\in\bigcup_{n=1}^{+\infty}F_n\right\}=\left(\bigcup_{n=1}^{+\infty}F_n\right)_x\in\mathcal{F}_2.$$

From the definition we get

$$\begin{aligned}y\in\left(\bigcup_{n=1}^{+\infty}F_n\right)_x &\iff (x,y)\in\bigcup_{n=1}^{+\infty}F_n\\ &\iff (x,y)\in F_n,\ \text{for some } n\in\mathbb{N}\\ &\iff y\in(F_n)_x,\ \text{for some } n\in\mathbb{N}\\ &\iff y\in\bigcup_{n=1}^{+\infty}(F_n)_x.\end{aligned}$$

Since $\mathcal{F}_n \in \mathcal{G}_x$ for any $n \in \mathbb{N}$, $(F_n)_x \in \mathcal{F}_2$ for any $n \in \mathbb{N}$. Hence $\bigcup_{n=1}^{+\infty} (F_n)_x \in \mathcal{F}_2$, and from the previous chain of implications we can conclude that $\left(\bigcup_{n=1}^{+\infty} F_n\right)_x \in \mathcal{F}_2$.

Finally, we show that $\mathcal{G} \subset \mathcal{G}_x$. If $E = E_1 \times E_2 \in \mathcal{G}$, then

$$E_x = \begin{cases} \emptyset, & x \notin E_1, \\ E_2, & x \in E_2, \end{cases}$$

which means $E_x \in \mathcal{F}_2$ and $E \in \mathcal{G}_x$.

Exercise 1.7. For $a, \sigma \in \mathbb{R}$ we have

$$\widehat{\gamma}(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{ix\xi} \exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\} dx.$$

We set $y = \frac{x-a}{\sigma}$, then $dx = \sigma dy$ and

$$\begin{aligned}
\widehat{\gamma}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i(\sigma y + a)\xi} e^{-y^2/2} dy = \frac{e^{ia\xi}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{y^2 - 2\sigma\xi iy}{2}\right\} dy \\
&= \frac{e^{ia\xi}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{y^2 - 2\sigma\xi iy - \xi^2\sigma^2 + \xi^2\sigma^2}{2}\right\} dy \\
&= e^{ia\xi - \frac{1}{2}\sigma^2\xi^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{(y - i\sigma\xi)^2}{2}\right\} dy \\
&= e^{ia\xi - \frac{1}{2}\sigma^2\xi^2}
\end{aligned}$$

Exercise 1.8.(Layer cake formula) Prove that if μ is a positive finite measure on $(X, \mathscr{F})$ and $0 \leq f \in L^1(X, \mu)$, then

$$\int_X f d\mu = \int_0^{+\infty} \mu(\{x \in X \mid f(x) > t\}) dt.$$

Proof. Let

$$H = \{(x,t) \in X \times [0, +\infty) : f(x) - t > 0\}$$

and observe that

$$\mu(\{x \in X \mid f(x) > t\}) = \mu(H^t) = \int_X \chi_{H^t}(x) d\mu(x)$$

where H^t is the vertical section of H (it is a measurable set by Exercise 1.5). It follows

$$\int_0^{+\infty} \mu(\{x \in X \mid f(x) > t\}) dt = \int_0^{+\infty} \left(\int_X \chi_{H^t}(x) d\mu(x)\right) dt =$$

now we apply Fubini's Theorem

$$= \int_X \left(\int_0^{+\infty} \chi_{H^t}(x) dt\right) d\mu(x) =$$

and by the fact that $\chi_{H^t}(x) = 1$ if, and only if, $|f(x)| > t$ we have

$$= \int_E \left(\int_0^{f(x)} dt \right) d\mu(x) =$$

we now use the fundamental theorem of calculus

$$= \int_X f(x) d\mu(x).$$