Lecture 1

Preliminaries

We present the basic notions of measure theory, with the aim of fixing the notation and making the exposition self-contained. We deal only with finite measures, even though of course positive infinite measures are the first natural examples. But, assuming that the basic notions relative to positive measures are known, we go straight to finite real measures because all measures we are going to discuss are so. In the first section we present real measures and the related notions of L^p space, absolute continuous and singular measure, the Radon-Nikodym theorem, weak convergence and product measures. In the case of topological spaces we introduce Borel and Radon measures. Next, we introduce characteristic functions (or Fourier transforms) of measures and Gaussian measures in $\mathbb{R}^d$.

1.1 Abstract Measure Theory

We start by introducing measurable spaces, i.e., sets equipped with a σ-algebra.

Definition 1.1.1 (σ-algebras and measurable spaces)**.** *Let X be a nonempty set and let $\mathscr{F}$ be a collection of subsets of X.*

(a) We say that $\mathscr{F}$ is an algebra *if $\emptyset \in \mathscr{F}$, $E_1 \cup E_2 \in \mathscr{F}$ and $X \setminus E_1 \in \mathscr{F}$ whenever E_1, $E_2 \in \mathscr{F}$.*

(b) We say that an algebra $\mathscr{F}$ is a σ-algebra *if for any sequence $(E_h) \subset \mathscr{F}$ its union $\bigcup_h E_h$ belongs to $\mathscr{F}$.*

(c) For any collection $\mathscr{G}$ of subsets of X, the σ-algebra generated by $\mathscr{G}$ is the smallest σ-algebra containing $\mathscr{G}$. If (X, τ) is a topological space, we denote by $\mathscr{B}(X)$ the σ-algebra of Borel subsets of X, i.e., the σ-algebra generated by the open subsets of X.

(d) If $\mathscr{F}$ is a σ-algebra in X, we call the pair $(X, \mathscr{F})$ a measurable space*.*

It is obvious by the De Morgan laws that algebras are closed under finite intersections, and σ-algebras under countable intersections. Moreover, since the intersection of any family of σ-algebras is a σ-algebra and the set of all subsets of X is a σ-algebra, the definition of generated σ-algebra is well posed. Once a σ-algebra has been fixed, it is possible to introduce positive measures.

Definition 1.1.2 (Finite measures)**.** *Let $(X, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, +\infty)$. We say that μ is a* positive finite measure *if $\mu(\emptyset) = 0$ and μ is σ-additive on $\mathscr{F}$, i.e., for any sequence (E_h) of pairwise disjoint elements of $\mathscr{F}$ the equality*

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h) \tag{1.1.1}$$

holds. We say that μ is a probability measure *if $\mu(X) = 1$.*
We say that $\mu : \mathscr{F} \to \mathbb{R}$ is a (finite) real measure *if $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are positive finite measures.*

If μ is a real measure, we define its *total variation* $|\mu|$ for every $E \in \mathscr{F}$ as follows:

$$|\mu|(E) := \sup\left\{\sum_{h=0}^{\infty} |\mu(E_h)| \colon E_h \in \mathscr{F} \text{ pairwise disjoint, } E = \bigcup_{h=0}^{\infty} E_h\right\} \tag{1.1.2}$$

and it turns out to be a positive measure, see Exercise 1.2. If μ is real, then (1.1.1) still holds, and the series converges absolutely, as the union is independent of the order. Notice also that the following equality holds:

$$|\mu|(\mathbb{R}^d) = \sup\Big\{\int_{\mathbb{R}^d} f d\mu : \ f \in C_b(\mathbb{R}^d), \|f\|_\infty \le 1\Big\}, \tag{1.1.3}$$

see Exercise 1.3.

Remark 1.1.3. (Monotonicity) Any positive finite measure μ is monotone with respect to set inclusion and continuous along monotone sequences, i.e., if (E_h) is an increasing sequence of sets (resp. a decreasing sequence of sets), then

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \lim_{h\to\infty} \mu(E_h), \qquad \text{resp.} \quad \mu\left(\bigcap_{h=0}^{\infty} E_h\right) = \lim_{h\to\infty} \mu(E_h),$$

see Exercise 1.1.

Definition 1.1.4 (Radon measures)**.** *A real measure μ on the Borel sets of a topological space X is called a* real Radon measure *if for every $B \in \mathscr{B}(X)$ and $\varepsilon > 0$ there is a compact set $K \subset B$ such that $|\mu|(B \setminus K) < \varepsilon$.*
A measure is tight *if the same property holds with $B = X$.*

Proposition 1.1.5. *If (X, d) is a separable complete metric space then every real measure on $(X, \mathscr{B}(X))$ is Radon.*

Proof. Observe that it is enough to prove the result for finite positive measures. The general case follows splitting the given real measure into its positive and negative parts. Let then μ be a positive finite measure on $(X, \mathscr{B}(X))$. Let us first show that it is a regular measure, i.e., for any $B \in \mathscr{B}(X)$ and for any $\varepsilon > 0$ there are an open set $G \supset B$ and a closed set $F \subset B$ such that $\mu(G \setminus F) < \varepsilon$. Indeed, for a given $\varepsilon > 0$, if $B = F$ is closed it suffices to consider open sets $G_\delta = \{x \in X : \ d(x, F) = \inf_{y \in F} d(x, y) < \delta\}$, getting $F = \bigcap_{\delta > 0} G_\delta$. As $\mu(G_\delta) \to \mu(F)$ as $\delta \to 0$, fixed $\varepsilon > 0$, for δ small enough by Remark 1.1.3 we have $\mu(G_\delta \setminus F) < \varepsilon$. Next, we show that the family $\mathscr{G}$ containing $\emptyset$ and all sets B such that for any $\varepsilon > 0$ there are an open set $G \supset B$ and a closed set $F \subset B$ such that $\mu(G \setminus F) < \varepsilon$ is a σ-algebra. To this aim, given a sequence $(B_n) \subset \mathscr{G}$, consider open sets G_n and closed sets F_n such that $F_n \subset B_n \subset G_n$ and $\mu(G_n \setminus F_n) < \varepsilon/2^{n+1}$. For $G = \cup_{n=1}^\infty G_n$ and $F = \cup_{n=1}^N F_n$, with $N \in \mathbb{N}$ such that $\mu(\cup_{n=1}^\infty F_n \setminus F) < \varepsilon/2$, we have $F \subset \cup_n B_n \subset G$ and $\mu(G \setminus F) < \varepsilon$. Therefore, $\mathscr{G}$ is closed under countable unions, and, since it is closed under complementation as well, it is a σ-algebra.
Since we have proved that all closed sets belong to $\mathscr{G}$, then it coincides with $\mathscr{B}(X)$.

As a consequence, we prove that any positive finite measure on $(X, \mathscr{B}(X))$ is Radon iff it is tight. If μ is Radon then it is tight by definition. Conversely, assuming that μ is tight, for every $\varepsilon > 0$ and every Borel set $B \subset X$, we may take a compact set K_1 such that $\mu(X \setminus K_1) < \varepsilon$ and a closed set $F \subset B$ such that $\mu(B \setminus F) < \varepsilon$. Then, defining the compact set $K := K_1 \cap F$ we have $\mu(B \setminus K) < 2\varepsilon$.

Therefore, to prove our statement it suffices to show that every Borel measure on X is tight. Let (x_n) be a dense sequence and notice that $X \subset \cup_{n=1}^\infty \overline{B}(x_n, 1/k)$ for every $k \in \mathbb{N}$. Then, given $\varepsilon > 0$, for every $k \in \mathbb{N}$ there is $N_k \in \mathbb{N}$ such that

$$\mu\Big(\bigcup_{n=1}^{N_k} \overline{B}(x_n, 1/k)\Big) > \mu(X) - \varepsilon/2^k.$$

Then, the compact set

$$K := \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{N_k} \overline{B}(x_n, 1/k)$$

verifies $\mu(K) > \mu(X) - \varepsilon$. □

Notice that property (1.1.3) holds in any metric space, but we shall use it only in $\mathbb{R}^d$.
Let us come to measurable functions.

Definition 1.1.6 (Measurable functions)**.** *Let $(X, \mathscr{F}, \mu)$ be a measure space and let Y be a topological space. A function $f : X \to Y$ is said to be $\mathscr{F}$-*measurable *(or μ-measurable) if $f^{-1}(A) \in \mathscr{F}$ for every open set $A \subset Y$.*
*If $(Y, \mathscr{G})$ is a measurable space, a function $f : X \to Y$ is said to be $\mathscr{F}$-*measurable *(or μ-measurable) if $f^{-1}(A) \in \mathscr{F}$ for every $A \in \mathscr{G}$.*

In particular, if f is $\mathscr{F}$-measurable then $f^{-1}(B) \in \mathscr{F}$ for every $B \in \mathscr{B}(Y)$. For $E \subset X$ we define the *indicator* (or *characteristic*) *function* of E, denoted by $\mathbb{1}_E$, by

$$\mathbb{1}_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

and we say that $f : X \to \mathbb{R}$ is a *simple function* if the image of f is finite, i.e., if f belongs to the vector space generated by the indicator functions. We assume that the readers are familiar with the usual notion of integral of a measurable function. We now define the L^p (semi)-norms and spaces as follows,

$$\|u\|_{L^p(X,\mu)} := \left(\int_X |u|^p \, d\mu \right)^{1/p}$$

if $1 \le p < \infty$, and

$$\|u\|_{L^\infty(X,\mu)} := \inf \{ C \in [0, +\infty] \colon |u(x)| \le C \text{ for } \mu\text{-a.e. } x \in X \} .$$

We define the space $L^p(X, \mu)$ as the space of equivalence classes of functions agreeing μ-a.e. such that $\|u\|_{L^p(X,\mu)} < \infty$. In this way, $\| \cdot \|_{L^p(X,\mu)}$ is a norm and $L^p(X, \mu)$ is a Banach space, see e.g. [D, Theorem 5.2.1]. When there is no risk of confusion, we use the shorter notation $\| \cdot \|_p$.

We assume that the reader is familiar also with the properties of integrals, measurable functions and L^p spaces as well as the main convergence theorems of Levi, Fatou, Lebesgue, see e.g. [D, Section 4.3]. We just recall the Lebesgue-Vitali theorem on uniformly integrable sequences, see Exercise 1.4.

Theorem 1.1.7 (Lebesgue-Vitali Convergence Theorem)**.** *Let $(X, \mathscr{F})$ be a measurable space, let μ be a positive finite measure on it and let (f_k) be a sequence of measurable functions such that*

$$\lim_{M \to \infty} \sup_{k \in \mathbb{N}} \int_{\{|f_k| > M\}} |f_k| \, d\mu = 0.$$

If $f_k \to f$ in measure, i.e.,

$$\lim_{k \to \infty} \mu(\{x \in X : \ |f_k(x) - f(x)| > \varepsilon\}) = 0 \qquad \text{for every } \varepsilon > 0 \tag{1.1.4}$$

and $|f(x)| < \infty$ μ-a.e., then $f \in L^1(X, \mu)$ and $\lim_{k \to \infty} \int_X |f - f_k| d\mu = 0$.

Given a σ-algebra, we have defined the class of measurable functions. Conversely, given a family of functions, it is possible to define a suitable σ-algebra.

Definition 1.1.8. *Given a family F of functions $f : X \to \mathbb{R}$, let us define the σ-algebra $\mathscr{E}(X, F)$ generated by F on X as the smallest σ-algebra such that all the functions $f \in F$ are measurable, i.e., the σ-algebra generated by the sets $\{f < t\}$, with $f \in F$ and $t \in \mathbb{R}$.*

Given a metric space, the set of real Borel measures μ is a vector space in an obviuos way. All continuous and bounded functions are in $L^1(X, \mu)$ and we define the *weak convergence of measures* by

$$\mu_j \to \mu \quad \Longleftrightarrow \quad \int_X f \, d\mu_j \to \int_X f \, d\mu \qquad \forall f \in C_b(X). \tag{1.1.5}$$

Let us now introduce the notions of absolute continuity and singularity of measures. Let μ be a positive finite measure and ν a real measure on the measurable space $(X, \mathscr{F})$. We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if $\mu(B) = 0 \implies |\nu|(B) = 0$ for every $B \in \mathscr{F}$. If μ, ν are real measures, we say that they are *mutually singular*, and write $\nu \perp \mu$, if there exists $E \in \mathscr{F}$ such that $|\mu|(E) = 0$ and $|\nu|(X \setminus E) = 0$. Notice that for mutually singular measures μ, ν the equality $|\mu + \nu| = |\mu| + |\nu|$ holds. If $\mu \ll \nu$ and $\nu \ll \mu$ we say that μ and ν are *equivalent* and write $\mu \approx \nu$. If μ is a positive measure and $f \in L^1(X, \mu)$, then the measure $\nu := f\mu$ defined below is absolutely continuous with respect to μ and the following integral representations hold, see Exercise 1.2:

$$\nu(B) = \int_B f \, d\mu, \quad |\nu|(B) = \int_B |f| \, d\mu \qquad \forall B \in \mathscr{F}. \tag{1.1.6}$$

In the following classical result we see that if a real measure ν is absolutely continuous with respect to μ, then the above integral representation holds, with a suitable f.

Theorem 1.1.9 (Radon-Nikodym)**.** *Let μ be a positive finite measure and let ν be a real measure. Then there is a unique pair of real measures ν^a, ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$. Moreover, there is a unique function $f \in L^1(X, \mu)$ such that $\nu^a = f\mu$. The function f is called the density (or Radon-Nikodym derivative) of ν with respect to μ and is denoted by $d\nu/d\mu$.*

Since trivially each real measure μ is absolutely continuous with respect to $|\mu|$, from the Radon-Nikodym theorem the *polar decomposition* of μ follows: there exists a unique real valued function $f \in L^1(X, |\mu|)$ such that $\mu = f|\mu|$ and $|f| = 1$ $|\mu|$-a.e.

The following result is a useful criterion of mutual singularity.

Theorem 1.1.10 (Hellinger)**.** *Let μ, ν be two probability measures on a measurable space $(X, \mathscr{F})$, and let λ be a positive measure such that $\mu \ll \lambda$, $\nu \ll \lambda$. Then the integral*

$$H(\mu, \nu) := \int_X \sqrt{\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}} \, d\lambda$$

is independent of λ and

$$2(1 - H(\mu, \nu)) \leq |\mu - \nu|(X) \leq 2\sqrt{1 - H(\mu, \nu)^2}. \tag{1.1.7}$$

Proof. Let us first take $\lambda = \mu + \nu$ and notice that $\mu, \nu \ll \lambda$. Then, setting $f := d\mu/d\lambda$ and $g := d\nu/d\lambda$, i.e., $\mu = f\lambda$ and $\nu = g\lambda$, we have $|\mu - \nu|(X) = \|f - g\|_{L^1(X,\lambda)}$ and integrating the inequalities

$$(\sqrt{f} - \sqrt{g})^2 \leq |f - g| = |\sqrt{f} - \sqrt{g}| \, |\sqrt{f} + \sqrt{g}|$$

we get

$$\begin{aligned}
\int_X (\sqrt{f} - \sqrt{g})^2 \, d\lambda = 2(1 - H(\mu, \nu)) &\leq \int_X |f - g| \, d\lambda = |\mu - \nu|(X) \\
&= \int_X |\sqrt{f} - \sqrt{g}| \, |\sqrt{f} + \sqrt{g}| \, d\lambda \\
&\leq \Big(\int_X |\sqrt{f} - \sqrt{g}|^2 \, d\lambda \Big)^{1/2} \Big(\int_X |\sqrt{f} + \sqrt{g}|^2 \, d\lambda \Big)^{1/2} \\
&= (2 - 2H(\mu, \nu))^{1/2} (2 + 2H(\mu, \nu))^{1/2} = 2\sqrt{1 - H(\mu, \nu)^2},
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. If λ' is another measure such that $\mu = f'\lambda' \ll \lambda'$ and $\nu = g'\lambda' \ll \lambda'$, then $\lambda \ll \lambda'$: setting $\phi := \frac{d\lambda}{d\lambda'}$, we have $f' = \phi f$, $g' = \phi g$ and then

$$\int_X \sqrt{\frac{d\mu}{d\lambda}\frac{d\nu}{d\lambda}}\, d\lambda = \int_X \sqrt{fg}\, d\lambda = \int_X \sqrt{fg}\phi\, d\lambda' = \int_X \sqrt{f'g'}\, d\lambda' = \int_X \sqrt{\frac{d\mu}{d\lambda'}\frac{d\nu}{d\lambda'}}\, d\lambda'.$$

□

Corollary 1.1.11. *If μ and ν are probability measures, then $\mu \perp \nu$ iff $H(\mu, \nu) = 0$.*

Proof. It is obvious from Hellinger's theorem that $|\mu-\nu|(X) = 2$ if and only if $H(\mu, \nu) = 0$. Let us show that this is equivalent to $\mu \perp \nu$. Using the notation in the proof of Theorem 1.1.10, notice that $H(\mu, \nu) = 0$ if and only if the set F defined by $F := \{fg \neq 0\}$ verifies $\lambda(F) = 0$ (hence also $\mu(F) = \nu(F) = 0$). Therefore, for the measurable set $E = \{f = 0, g > 0\}$ we have $\mu(E) = \nu(X \setminus E) = 0$ and the thesis follows. □

We recall the notions of *push-forward* of a measure (or *image measure*) and the constructions and main properties of *product measure*. The push-forward of a measure generalises the classical change of variable formula.

Definition 1.1.12 (Push-forward)**.** *Let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be measurable spaces, and let $f : X \to Y$ be such that $f^{-1}(F) \in \mathscr{F}$ whenever $F \in \mathscr{G}$. For any positive or real measure μ on $(X, \mathscr{F})$ we define the push-forward measure or the law of μ under f $\mu \circ f^{-1}$, sometimes denoted by $f_{\#}\mu$, in $(Y, \mathscr{G})$ by*

$$\mu \circ f^{-1}(F) := \mu\left(f^{-1}(F)\right) \qquad \forall F \in \mathscr{G}.$$

The *change of variables formula* immediately follows from the previous definition. If $u \in L^1(Y, \mu \circ f^{-1})$, then $u \circ f \in L^1(X, \mu)$ and we have the equality

$$\int_Y u\, d(\mu \circ f^{-1}) = \int_X (u \circ f)\, d\mu. \tag{1.1.8}$$

The above relation is nothing but the definition for simple functions, and is immediately extended to the whole of L^1 by density.

We consider now two measure spaces and describe the natural resulting structure on their cartesian product.

Definition 1.1.13 (Product σ-algebra)**.** *Let $(X_1, \mathscr{F}_1)$ and $(X_2, \mathscr{F}_2)$ be measure spaces. The* product σ-algebra *of $\mathscr{F}_1$ and $\mathscr{F}_2$, denoted by $\mathscr{F}_1 \times \mathscr{F}_2$, is the σ-algebra generated in $X_1 \times X_2$ by*

$$\mathscr{G} = \{E_1 \times E_2 \colon E_1 \in \mathscr{F}_1, E_2 \in \mathscr{F}_2\}.$$

Remark 1.1.14. Let $E \in \mathscr{F}_1 \times \mathscr{F}_2$; then for every $x \in X_1$ the section $E_x := \{y \in X_2 \colon (x, y) \in E\}$ belongs to $\mathscr{F}_2$, and for every $y \in X_2$ the section $E^y := \{x \in X_1 \colon (x, y) \in E\}$ belongs to $\mathscr{F}_1$. In fact, it is easily checked that the families

$$\mathscr{G}_x := \{F \in \mathscr{F}_1 \times \mathscr{F}_2 \colon F_x \in \mathscr{F}_2\}, \quad \mathscr{G}^y := \{F \in \mathscr{F}_1 \times \mathscr{F}_2 \colon F^y \in \mathscr{F}_1\}$$

are σ-algebras in $X_1 \times X_2$ and contain $\mathscr{G}$ see Exercise 1.5.

Theorem 1.1.15 (Fubini)**.** *Let $(X_1, \mathscr{F}_1, \mu_1)$, $(X_2, \mathscr{F}_2, \mu_2)$ be measure spaces with μ_1, μ_2 positive and finite. Then, there is a unique positive finite measure μ on $(X_1 \times X_2, \mathscr{F}_1 \times \mathscr{F}_2)$, denoted also by $\mu_1 \otimes \mu_2$, such that*

$$\mu(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2) \qquad \forall E_1 \in \mathscr{F}_1, \forall E_2 \in \mathscr{F}_2.$$

Furthermore, for any μ-measurable function $u : X_1 \times X_2 \to [0, \infty]$ the functions

$$x \mapsto \int_{X_2} u(x, y)\, \mu_2(dy) \quad \text{and} \quad y \mapsto \int_{X_1} u(x, y)\, \mu_1(dx)$$

are respectively μ_1-measurable and μ_2-measurable and

$$\begin{aligned}\int_{X_1 \times X_2} u\, d\mu &= \int_{X_1} \left(\int_{X_2} u(x, y)\, \mu_2(dy) \right) \mu_1(dx) \\ &= \int_{X_2} \left(\int_{X_1} u(x, y)\, \mu_1(dx) \right) \mu_2(dy).\end{aligned}$$

Remark 1.1.16. More generally, it is possible to construct a product measure on infinite cartesian products. If I is a set of indices, typically $I = [0, 1]$ or $I = \mathbb{N}$, and $(X_t, \mathscr{F}_t, \mu_t)$, $t \in I$, is a family of measure spaces, the product σ-algebra is that generated by the family of sets of the form

$$B = B_1 \times \cdots \times B_n \times \bigtimes_{t \in I \setminus \{t_1, \ldots, t_k\}} X_t, \quad B_k \in \mathscr{F}_{t_k},$$

whose measure is $\mu(B) = \mu_{t_1}(B_1) \cdots \mu_{t_n}(B_n)$.

In the sequel we shall sometimes encounter some ideas coming from probability theory and stochastic analysis. In order to simplify several computations concerning probability measures on $\mathbb{R}^d$, it is often useful to use *characteristic functions* of measures. This is the probabilistic counterpart of Fourier transform. Indeed, given a probability measure μ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, we define its characteristic function by setting

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{ix \cdot \xi}\, \mu(dx), \qquad \xi \in \mathbb{R}^d. \tag{1.1.9}$$

We list the main elementary properties of characteristic functions, whose proofs are in Exercise 1.6.

1. $\hat{\mu}$ is uniformly continuous on $\mathbb{R}^d$;
2. $\hat{\mu}(0) = 1$;
3. if $\hat{\mu}_1 = \hat{\mu}_2$ then $\mu_1 = \mu_2$;
4. if $\mu_j \to \mu$ in the sense of (1.1.5), then $\hat{\mu}_j \to \hat{\mu}$ uniformly on compacts;
5. if (μ_j) is a sequence of probability measures and there is $\phi : \mathbb{R}^d \to \mathbb{C}$ continuous in $\xi = 0$ such that $\hat{\mu}_j \to \phi$ pointwise, then there is a probability measure μ such that $\hat{\mu} = \phi$.

1.2 Gaussian measures

Gaussian (probability) measures are the main reference measures we shall encounter in the Lectures. Let us start from the finite dimensional case. We recall the following elementary equality

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}}\exp\Big\{-\frac{(x-a)^2}{2\sigma^2}\Big\}\,dx=1 \tag{1.2.1}$$

that holds for all $a\in\mathbb{R}$ and $\sigma>0$. An easy way to prove (1.2.1) is to compute the double integral

$$\int_{\mathbb{R}^2}\exp\{-(x^2+y^2)\}\,dxdy$$

in polar coordinates and apply Fubini Theorem 1.1.15 and the change of variables formula.

Definition 1.2.1 (Gaussian measures on $\mathbb{R}$)**.** *A probability measure γ on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ is called* Gaussian *if it is either a Dirac measure δ_a at a point a (in this case, we put $\sigma=0$), or a measure absolutely continuous with respect to the Lebesgue measure λ_1 with density*

$$\frac{1}{\sigma\sqrt{2\pi}}\exp\Big\{-\frac{(x-a)^2}{2\sigma^2}\Big\}.$$

In this case we call a the mean, *σ the* mean-square deviation *and σ^2 the* variance *of γ and we say that γ is* centred *or* symmetric *if $a=0$ and* standard *if $\sigma=1$.*

By elementary computations we get

$$a=\int_{\mathbb{R}}x\,\gamma(dx),\qquad \sigma^2=\int_{\mathbb{R}}(x-a)^2\,\gamma(dx).$$

Remark 1.2.2. For every $a,\sigma\in\mathbb{R}$ we have $\hat\gamma(\xi)=e^{ia\xi-\frac12\sigma^2\xi^2}$, see Exercise 1.7. Conversely, by property 3 of characteristic functions, a probability measure on $\mathbb{R}$ is Gaussian iff its characteristic function has this form. Therefore, it is easy to recognise a Gaussian measure from its characteristic function. This is true in $\mathbb{R}^d$, as we are going to see in Proposition 1.2.4, and also in infinite dimensions, as we shall see in the next Lecture.

Let us come to Gaussian measures in $\mathbb{R}^d$.

Definition 1.2.3 (Gaussian measures on $\mathbb{R}^d$)**.** *A probability measure γ on $\mathbb{R}^d$ is said to be Gaussian if for every linear functional ℓ on $\mathbb{R}^d$ the measure $\gamma\circ\ell^{-1}$ is Gaussian on $\mathbb{R}$.*

The first example of Gaussian measure in $\mathbb{R}^d$ is $\gamma_d:=(2\pi)^{-d/2}e^{-|x|^2}\lambda_d$, that is called *standard Gaussian measure.* We denote by G_d the *standard Gaussian density* in $\mathbb{R}^d$, i.e., the density of γ_d with respect to λ_d. Notice also that if $d=h+k$ then $\gamma_d=\gamma_h\otimes\gamma_k$.

The following result gives an useful characterisation of a Gaussian measure through its characteristic function.

Proposition 1.2.4. *A measure γ on $\mathbb{R}^d$ is Gaussian if and only if its characteristic function is*

$$\hat{\gamma}(\xi) = \exp\Big\{ia \cdot \xi - \frac{1}{2}Q\xi \cdot \xi\Big\} \tag{1.2.2}$$

for some $a \in \mathbb{R}^d$ and Q nonnegative $d \times d$ symmetric matrix. Moreover, γ is absolutely continuous with respect to the Lebesgue measure λ_d if and only if Q is nondegenerate. In this case, the density of γ is

$$\frac{1}{\sqrt{(2\pi)^d \det Q}} \exp\Big\{-\frac{1}{2}\big(Q^{-1}(x-a)\cdot(x-a)\big)\Big\}. \tag{1.2.3}$$

Proof. Let γ be a measure such that (1.2.2) holds. Then, for every linear funtional $\ell : \mathbb{R}^d \to \mathbb{R}$ (here we identify ℓ with the vector in $\mathbb{R}^d$ such that $\ell(x) = \ell \cdot x$) we may compute the characteristic function of the measure $\mu_\ell := \gamma \circ \ell^{-1}$ on $\mathbb{R}$:

$$\widehat{\mu_\ell}(\tau) = \int_{\mathbb{R}} e^{i\tau t}\, \mu_\ell(dt) = \int_{\mathbb{R}^d} e^{i\tau\ell(x)}\, \gamma(dx) = \hat{\gamma}(\tau\ell) = \exp\Big\{i\tau a \cdot \ell - \frac{\tau^2}{2}Q\ell \cdot \ell\Big\}$$

by (1.2.2). Therefore, by Remark 1.2.2 μ_ℓ is a Gaussian measure with mean $a_\ell = a \cdot \ell$ and variance $\sigma_\ell^2 = Q\ell \cdot \ell$, and also γ is a Gaussian measure by the arbitrariness of ℓ.

Conversely, assume that μ_ℓ is Gaussian for every ℓ as above. Its mean a_ℓ and its variance σ_ℓ^2 are given by

$$a_\ell := \int_{\mathbb{R}} t\, \mu_\ell(dt) = \int_{\mathbb{R}^d} \ell(x)\, \gamma(dx) = \ell \cdot \Big(\int_{\mathbb{R}^d} x\, \gamma(dx)\Big) \tag{1.2.4}$$

$$\sigma_\ell^2 := \int_{\mathbb{R}^d} (t - a_\ell)^2\, \mu_\ell(dt) = \int_{\mathbb{R}^d} (\ell(x) - a_\ell)^2\, \gamma(dx). \tag{1.2.5}$$

These formulas show that the map $\ell \mapsto a_\ell$ is linear and the map $\ell \mapsto \sigma_\ell^2$ is a nonnegative quadratic form. Therefore, there are a vector $a \in \mathbb{R}^d$ and a nonnegative definite symmetric matrix $Q = (Q_{ij})$ such that $a_\ell = a \cdot \ell$ and $\sigma_\ell^2 = Q\ell \cdot \ell$, whence (1.2.2) follows. Notice that

$$a = \int_{\mathbb{R}^d} x\, \gamma(dx), \qquad Q_{ij} = \int_{\mathbb{R}^d} (x_i - a_i)(x_j - a_j)\, \gamma(dx).$$

To prove the last part of the statement, let us assume that $\gamma \ll \lambda_d$, i.e. $\gamma = f\lambda_d$. We want to show that $Q\ell \cdot \ell = 0$ iff $\ell = 0$. From (1.2.4), (1.2.5) we have

$$Q\ell \cdot \ell = \int_{\mathbb{R}^d} (\ell \cdot (x-a))^2 f(x) dx,$$

then $Q\ell \cdot \ell = 0$ iff $(\ell \cdot (x-a))^2 = 0$ for a.e. $x \in \mathbb{R}^d$, i.e. iff $\ell = 0$, as $f \not\equiv 0$. Hence Q is nondegenerate.
Viceversa, if Q is nondegenerate, we consider the measure $\nu = f\lambda_d$ with f given by (1.2.3) and we compute its characteristic function. Using the change of variable $z = Q^{-1/2}(x-a)$,

since $\gamma_d = \otimes_{j=1}^d \gamma_1$, we have:

$$
\begin{aligned}
\hat{\nu}(\xi) &= \int_{\mathbb{R}^d} \exp\{i\xi\cdot x\} f(x)dx = \frac{1}{\sqrt{(2\pi)^d \det Q}} \int_{\mathbb{R}^d} \exp\Big\{ i\xi\cdot x - \frac{1}{2}\big(Q^{-1}(x-a)\cdot(x-a)\big)\Big\} dx \\
&= \exp\{i\xi\cdot a\} \int_{\mathbb{R}^d} \exp\{iQ^{1/2}\xi\cdot z\}\gamma_d(dz) = \exp\{i\xi\cdot a\} \prod_{j=1}^d \int_{\mathbb{R}} \exp\{i(Q^{1/2}\xi)_j t\}\gamma_1(dt) \\
&= \exp\{i\xi\cdot a\} \prod_{j=1}^d \hat{\gamma}_1((Q^{1/2}\xi)_j) = \exp(i\xi\cdot a) \prod_{j=1}^d \exp\left\{-\frac{1}{2}((Q^{1/2}\xi)_j)^2\right\} \\
&= \exp\left\{ i\xi\cdot a - \frac{1}{2}Q\xi\cdot\xi\right\} = \hat{\gamma}(\xi).
\end{aligned}
$$

Hence, by property 3 of the characteristic function, $\gamma = \nu$. □

Remark 1.2.5. If γ is a Gaussian measure and (1.2.2) holds, we call a the *mean* and Q the *covariance* of γ, and we write $\gamma = \mathscr{N}(a,Q)$ when it is useful to emphasise the relevant parameters. If the matrix Q is invertible then the Gaussian measure $\gamma = \mathscr{N}(a,Q)$ is said to be *nondegenerate*. Its density, given by (1.2.3), is denoted $G_{a,Q}$. The nondegeneracy is equivalent to the fact that $\gamma_\ell \ll \lambda_1$ for every $\ell \in \mathbb{R}^d$.

Proposition 1.2.6. *Every centred Gaussian measure γ on $\mathbb{R}^d$ is invariant under the rotation map ϕ defined, for every $\theta \in \mathbb{R}$, by $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by $\phi(x,y) := \sin\theta x + \cos\theta y$; then, the image measure $(\gamma\otimes\gamma)\circ\phi^{-1}$ in $\mathbb{R}^d$ is γ.*

Proof. We use characteristic functions and Proposition 1.2.4. Indeed, the characteristic function of γ is $\exp\{-\frac{1}{2}Q\xi\cdot\xi\}$ for some nonnegative $d\times d$ matrix Q. Then we may compute the characteristic function of $\mu := \gamma\circ\phi^{-1}$ as follows:

$$
\begin{aligned}
\hat{\mu}(\xi) &= \int_{\mathbb{R}^d} e^{iz\xi}\,\mu(dz) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(\sin\theta x + \cos\theta y)\xi}\,\gamma\otimes\gamma(d(x,y)) \\
&= \int_{\mathbb{R}^d} e^{i(\sin\theta x)\xi}\,\gamma(dx) \int_{\mathbb{R}^d} e^{i(\cos\theta y)\xi}\,\gamma(dy) \\
&= \exp\{-\frac{1}{2}\sin^2\theta Q\xi\cdot\xi\}\exp\{-\frac{1}{2}\cos^2\theta Q\xi\cdot\xi\} \\
&= \exp\{-\frac{1}{2}Q\xi\cdot\xi\},
\end{aligned}
$$

and the thesis follows from property 3 of characteristic functions. □

Remark 1.2.7. We point out that the property stated in Proposition 1.2.6 is not the invariance of γ under rotations in $\mathbb{R}^d$. Indeed, rotation invariance holds iff the covariance of γ is a positive multiple of an orthogonal matrix.

1.3 Exercises

Exercise 1.1. Let μ be a positive finite measure on $(X, \mathscr{F})$. Prove the monotonicity properties stated in Remark 1.1.3.

Exercise 1.2. Prove that the set function $|\mu|$ defined in (1.1.2) is a positive finite measure and that the integral representation for $|\nu| = |f\mu|$ in (1.1.6) holds. Prove also that if $\mu \perp \nu$ then $|\mu + \nu| = |\mu| + |\nu|$.

Exercise 1.3. Prove the equality (1.1.3).

Exercise 1.4. Prove the Vitali–Lebesgue Theorem 1.1.7.

Exercise 1.5. Prove that the families $\mathscr{G}_x$ and $\mathscr{G}^y$ defined in Remark 1.1.14 are σ-algebras.

Exercise 1.6. Prove the properties of characteristic functions listed in Section 1.1.

Exercise 1.7. Prove the equality $\hat{\gamma}(\xi) = e^{ia\xi - \frac{1}{2}\sigma^2\xi^2}$ stated in Remark 1.2.2.

Exercise 1.8. (Layer cake formula) Prove that if μ is a positive finite measure on $(X, \mathscr{F})$ and $0 \le f \in L^1(X, \mu)$ then

$$\int_X f \, d\mu = \int_0^\infty \mu\big(\{x \in X : \ f(x) > t\}\big) \, dt.$$

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